

**ON THE STRONGLY AMBIGUOUS CLASSES OF  $\mathbb{k}/\mathbb{Q}(i)$   
WHERE  $\mathbb{k} = \mathbb{Q}(\sqrt{2p_1p_2}, i)$**

ABDELMALEK AZIZI, ABDELKADER ZEKHNINI, AND MOHAMMED TAOUS

**ABSTRACT.** We construct an infinite family of imaginary bicyclic biquadratic number fields  $\mathbb{k}$  with the 2-ranks of their 2-class groups are  $\geq 3$ , whose strongly ambiguous classes of  $\mathbb{k}/\mathbb{Q}(i)$  capitulate in the absolute genus field  $\mathbb{k}^{(*)}$ , which is strictly included in the relative genus field  $(\mathbb{k}/\mathbb{Q}(i))^*$  and we study the capitulation of the 2-ideal classes of  $\mathbb{k}$  in its quadratic extensions included in  $\mathbb{k}^{(*)}$ .

### 1. Introduction.

Let  $\mathbb{k}$  be an algebraic number field and let  $\mathbf{C}_{\mathbb{k},2}$  denote its 2-class group, that is the 2-Sylow subgroup of the ideal class group,  $\mathbf{C}_{\mathbb{k}}$ , of  $\mathbb{k}$ . We denote by  $\mathbb{k}^{(*)}$  the absolute genus field of  $\mathbb{k}$ . Suppose  $\mathbb{F}$  is a finite extension of  $\mathbb{k}$ , then we say that an ideal class of  $\mathbb{k}$  capitulates in  $\mathbb{F}$  if it is in the kernel of the homomorphism

$$J_{\mathbb{F}} : \mathbf{C}_{\mathbb{k}} \longrightarrow \mathbf{C}_{\mathbb{F}}$$

induced by extension of ideals from  $\mathbb{k}$  to  $\mathbb{F}$ . An important problem in Number Theory is to determine explicitly the kernel of  $J_{\mathbb{F}}$ , which is usually called the capitulation kernel. The classical Principal Ideal Theorem asserts that  $\ker J_{\mathbb{F}}$  is all  $\mathbf{C}_{\mathbb{k}}$  if  $\mathbb{F}$  is the Hilbert class field of  $\mathbb{k}$ . If  $\mathbb{F}$  is the relative genus field of a cyclic extension  $\mathbb{K}/\mathbb{k}$ , which we denote by  $(\mathbb{K}/\mathbb{k})^*$  and that is the maximal unramified extension of  $\mathbb{K}$  which is obtained by composing  $\mathbb{K}$  and an abelian extension over  $\mathbb{k}$ , F. Terada states in [8] that all the ambiguous ideal classes of  $\mathbb{K}/\mathbb{k}$  capitulate in  $(\mathbb{K}/\mathbb{k})^*$ ; if  $\mathbb{F}$  is the absolute genus field of an abelian extension  $\mathbb{K}/\mathbb{Q}$ , then H. Furuya confirms in [9] that every strongly ambiguous class, that is an ambiguous ideal class represented by an ambiguous ideal, of  $\mathbb{K}/\mathbb{Q}$  capitulate in  $\mathbb{F}$ . In this paper we construct a family of number field  $\mathbb{k}$  for which all the strongly ambiguous classes of  $\mathbb{k}/\mathbb{Q}(i)$  capitulate in  $\mathbb{k}^{(*)} \subsetneq (\mathbb{k}/\mathbb{Q}(i))^*$  and all classes that capitulate in an unramified quadratic extension  $\mathbb{K}$  of  $\mathbb{k}$  that is abelian over  $\mathbb{Q}$  are strongly ambiguous classes of  $\mathbb{k}/\mathbb{Q}(i)$ .

---

2010 *Mathematics Subject Classification.* 11R11, 11R16, 11R20, 11R27, 11R29, 11R37.

*Key words and phrases.* absolute and relative genus fields, fundamental systems of units, 2-class group, capitulation, quadratic fields, biquadratic fields, multiquadratic CM-fields, Hilbert class fields.

Let  $p_1 \equiv p_2 \equiv 1 \pmod{4}$  be primes,  $\mathbb{k} = \mathbb{Q}(\sqrt{2p_1p_2}, i)$  and  $\mathbb{K}$  be an unramified quadratic extension of  $\mathbb{k}$  that is abelian over  $\mathbb{Q}$ , let  $Am_s(\mathbb{k}/\mathbb{Q}(i))$  denote the group of the strongly ambiguous classes of  $\mathbb{k}/\mathbb{Q}(i)$ . Our object is to determine the 2-classes of the field  $\mathbb{k}$  which capitulate in the extension  $\mathbb{K}$  involving the fundamental units of the three quadratic subfields of  $\mathbb{k}$ , we prove that  $\ker J_{\mathbb{K}} \subset Am_s(\mathbb{k}/\mathbb{Q}(i))$  and we infer that  $Am_s(\mathbb{k}/\mathbb{Q}(i)) \subseteq \ker J_{\mathbb{K}^{(*)}}$ . As an application we will determine these 2-classes when  $\mathbf{C}_{\mathbb{k},2}$  is of type  $(2, 2, 2)$ . This study is based on genus theory, class group theory and other theorems as the next result giving the number of classes which capitulate in a cyclic extension of prime degree: if  $K/k$  is a cyclic extension of prime degree, then the number of classes which capitulate in  $K/k$  is:

$$[K : k][E_k : N(E_K)], \quad (1)$$

where  $E_k$  and  $E_K$  are the unit groups of  $k$  and  $K$  respectively and  $N$  is the norm of  $K/k$ .

Let  $m$  be a square-free integer and  $K$  be a number field, during this paper, we adopt the following notations:

- $p_1 \equiv p_2 \equiv 1 \pmod{4}$  are primes.
- $\mathbb{k}$ : denotes the field  $\mathbb{Q}(\sqrt{2p_1p_2}, \sqrt{-1})$ .
- $\mathcal{O}_K$ : the ring of integers of  $K$ .
- $E_K$ : the unit group of  $\mathcal{O}_K$ .
- $W_K$ : the group of roots of unity contained in  $K$ .
- $F.S.U$ : the fundamental system of units.
- $K^+$ : the maximal real subfield of  $K$ , if it is a CM-field.
- $Q_K = [E_K : W_K E_{K^+}]$  is Hasse's unit index, if  $K$  is a CM-field.
- $q(K/\mathbb{Q}) = [E_K : \prod_i^s E_{k_i}]$  is the unit index of  $K$ , if  $K$  is multiquadratic, where  $k_i$  are the quadratic subfields of  $K$ .
- $K^{(*)}$ : the absolute genus field of  $K$ .
- $\mathbf{C}_{K,2}$ : the 2-class group of  $K$ .
- $i = \sqrt{-1}$ .
- $\varepsilon_m$ : the fundamental unit of  $\mathbb{Q}(\sqrt{m})$ .
- $N(a)$ : denotes the absolute norm of a number  $a$ .

Our main theorem is.

**Theorem.** *Let  $Am_s(\mathbb{k}/\mathbb{Q}(i))$  denote the group of the strongly ambiguous classes of  $\mathbb{k}/\mathbb{Q}(i)$ . If  $\mathbb{K}$  is an unramified quadratic extension of  $\mathbb{k}$  that is abelian over  $\mathbb{Q}$ , then*

- (1)  $\ker J_{\mathbb{K}} \subset Am_s(\mathbb{k}/\mathbb{Q}(i))$ .
- (2)  $Am_s(\mathbb{k}/\mathbb{Q}(i)) \subseteq \ker J_{\mathbb{K}^{(*)}}$ .

The proof of this theorem is based on several results of units, the class-group of  $\mathbb{k}$  and its subgroup of the strongly ambiguous classes.

2. *F.S.U* OF SOME CM-FIELDS

Let us first collect some results that will be useful in what follows.

Let  $m$  and  $n$  be two positive square-free integers, such that  $(m, n) = 1$ ; let  $\varepsilon_1$  (resp.  $\varepsilon_2, \varepsilon_3$ ) denote the fundamental unit of  $\mathbb{Q}(\sqrt{m})$  (resp.  $\mathbb{Q}(\sqrt{n}), \mathbb{Q}(\sqrt{mn})$ ). Put  $K_0 = \mathbb{Q}(\sqrt{m}, \sqrt{n})$ ,  $K = K_0(i)$ .

Put  $B = \{\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_1\varepsilon_2, \varepsilon_1\varepsilon_3, \varepsilon_2\varepsilon_3, \varepsilon_1\varepsilon_2\varepsilon_3\}$  and  $B' = B \cup \{\sqrt{\mu}/\mu \in B \text{ and } \sqrt{\mu} \in K_0\}$ , then a *F.S.U* of  $K_0$  is a system consisting of three elements chosen from  $B'$  (see [11] for details).

To determine a *F.S.U* of  $K$ , we will use the following result [1, p.18] that the first author deduced from a theorem of Hasse [10, §21, Satz 15].

**Lemma 1.** *Let  $n \geq 2$  be an integer and  $\xi_n$  a  $2^n$ -th primitive root of unity, then*

$$\xi_n = \frac{1}{2}(\mu_n + \lambda_n i), \quad \text{where} \quad \mu_n = \sqrt{2 + \mu_{n-1}}, \quad \lambda_n = \sqrt{2 - \mu_{n-1}},$$

$$\mu_2 = 0, \lambda_2 = 2 \quad \text{and} \quad \mu_3 = \lambda_3 = \sqrt{2}.$$

Let  $n_0$  be the greatest integer such that  $\xi_{n_0}$  is contained in  $K$ ,  $\{\varepsilon'_1, \varepsilon'_2, \varepsilon'_3\}$  a *F.S.U* of  $K_0$  and  $\varepsilon$  a unit of  $K_0$  such that  $(2 + \mu_{n_0})\varepsilon$  is a square in  $K_0$  (if it exists). Then a *F.S.U* of  $K$  is one of the following systems:

- (a)  $\{\varepsilon'_1, \varepsilon'_2, \varepsilon'_3\}$  if  $\varepsilon$  does not exist;
- (b)  $\{\varepsilon'_1, \varepsilon'_2, \sqrt{\xi_{n_0}\varepsilon}\}$  if  $\varepsilon$  exists; in this case  $\varepsilon = \varepsilon_1^{i_1} \varepsilon_2^{i_2} \varepsilon_3^{i_3}$ , where  $i_1, i_2 \in \{0, 1\}$  (up to a permutation).

**Lemma 2.** *If  $\varepsilon_1, \varepsilon_2$  and  $\varepsilon_3$  have negative norms, then*

- (1) *If  $\varepsilon_1\varepsilon_2\varepsilon_3$  is a square in  $K_0$ , then  $\{\varepsilon_1, \varepsilon_2, \sqrt{\varepsilon_1\varepsilon_2\varepsilon_3}\}$  is a *F.S.U* of  $K_0$  and  $Q_K = 1$ .*
- (2) *If  $\varepsilon_1\varepsilon_2\varepsilon_3$  is not a square in  $K_0$ , then  $\{\varepsilon_1, \varepsilon_2, \varepsilon_3\}$  is a *F.S.U* of  $K_0$  and  $Q_K = 2$  if and only if  $2\varepsilon_1\varepsilon_2\varepsilon_3$  is a square in  $K_0$ .*
- (3) *If  $Q_K = 2$ , then  $\{\varepsilon_1, \varepsilon_2, \sqrt{i\varepsilon_1\varepsilon_2\varepsilon_3}\}$  is a *F.S.U* of  $K$ .*

*Proof.* See Propositions 15, 16 of [4]. □

**Lemma 3** ([2], Lemma 7). *Let  $p$  be an odd prime and  $\varepsilon_{2p} = x + y\sqrt{2p}$ . If  $N(\varepsilon) = 1$ , then  $x \pm 1$  is a square in  $\mathbb{N}$  and  $2\varepsilon$  is a square in  $\mathbb{Q}(\sqrt{2p})$ .*

**Lemma 4** ([2], Lemma 5). *Let  $d$  be a square-free integer and  $\varepsilon_d = x + y\sqrt{d}$ , where  $x, y$  are integers or semi-integers. If  $N(\varepsilon) = 1$ , then  $2(x \pm 1)$  and  $2d(x \pm 1)$  are not squares in  $\mathbb{Q}$ .*

**Lemma 5** ([1], 3.(1) p.19). *Let  $d$  be a square-free integer, different from 2 and  $k = \mathbb{Q}(\sqrt{d}, i)$ , put  $\varepsilon_d = x + y\sqrt{d}$ .*

- (i) *If  $N(\varepsilon_d) = -1$ , then  $\{\varepsilon_d\}$  is a F.S.U of  $k$ .*
- (ii) *If  $N(\varepsilon_d) = 1$ , then  $\{\sqrt{i\varepsilon_d}\}$  is a F.S.U of  $k$  if and only if  $x \pm 1$  is a square in  $\mathbb{N}$  (i.e.  $2\varepsilon_d$  is a square in  $\mathbb{Q}(\sqrt{d})$ ). Else  $\{\varepsilon_d\}$  is a F.S.U of  $k$  (this result is also in [15]).*

**Lemma 6** ([5], Corollary 3.2). *Let  $d$  be square-free integer and  $k = \mathbb{Q}(\sqrt{d}, i)$ , then  $Q_k = 1$  if one of the following conditions holds:*

- (i)  $d \equiv 1 \pmod{4}$ .
- (ii) *There exists an integer  $d'$  dividing  $d$  such that  $d' \equiv 5 \pmod{8}$ .*

**2.1. F.S.U of the field  $\mathbb{K} = \mathbb{Q}(\sqrt{p_1}, \sqrt{2p_2}, i)$ .** Let  $\mathbb{K} = \mathbb{k}(\sqrt{p_1}) = \mathbb{Q}(\sqrt{p_1}, \sqrt{2p_2}, i)$ . We denote by  $\varepsilon_1$  (resp.  $\varepsilon_2, \varepsilon_3$ ) the fundamental unit of  $\mathbb{Q}(\sqrt{p_1})$  (resp.  $\mathbb{Q}(\sqrt{2p_2}), \mathbb{Q}(\sqrt{2p_1p_2})$ ) and put  $\varepsilon_3 = x + y\sqrt{2p_1p_2}$ . The purpose of this sub-paragraph is to establish the following theorem.

**Theorem 1.** *Keep the notations previously mentioned, then*

- (1) *If  $N(\varepsilon_2) = N(\varepsilon_3) = -1$ , then*
  - (i) *If  $\varepsilon_1\varepsilon_2\varepsilon_3$  is a square in  $\mathbb{K}^+$ , then  $\{\varepsilon_1, \varepsilon_2, \sqrt{\varepsilon_1\varepsilon_2\varepsilon_3}\}$  is a F.S.U of  $\mathbb{K}^+$ ,  $\mathbb{K}$  and  $Q_{\mathbb{K}} = 1$ .*
  - (ii) *Else  $\{\varepsilon_1, \varepsilon_2, \varepsilon_3\}$  is a F.S.U of  $\mathbb{K}^+$  and that of  $\mathbb{K}$  is  $\{\varepsilon_1, \varepsilon_2, \sqrt{i\varepsilon_1\varepsilon_2\varepsilon_3}\}$  and  $Q_{\mathbb{K}} = 2$ .*
- (2) *If  $N(\varepsilon_2) = -N(\varepsilon_3) = 1$ , then the F.S.U of  $\mathbb{K}^+$  is  $\{\varepsilon_1, \varepsilon_2, \varepsilon_3\}$  and that of  $\mathbb{K}$  is  $\{\varepsilon_1, \sqrt{i\varepsilon_2}, \varepsilon_3\}$  and  $Q_{\mathbb{K}} = 2$ .*
- (3) *If  $N(\varepsilon_3) = -N(\varepsilon_2) = 1$ , then*
  - (i) *If  $2p_1(x \pm 1)$  is a square in  $\mathbb{N}$ , then  $\{\varepsilon_1, \varepsilon_2, \sqrt{\varepsilon_3}\}$  is a F.S.U of  $\mathbb{K}^+$ ,  $\mathbb{K}$  and  $Q_{\mathbb{K}} = 1$ .*
  - (ii) *Else  $\{\varepsilon_1, \varepsilon_2, \varepsilon_3\}$  is a F.S.U of  $\mathbb{K}^+$  and that of  $\mathbb{K}$  is  $\{\varepsilon_1, \varepsilon_2, \sqrt{i\varepsilon_3}\}$  and  $Q_{\mathbb{K}} = 2$ .*
- (4) *If  $N(\varepsilon_3) = N(\varepsilon_2) = 1$ , then*
  - (i) *If  $2p_1(x \pm 1)$  is a square in  $\mathbb{N}$ , then  $\{\varepsilon_1, \varepsilon_2, \sqrt{\varepsilon_3}\}$  is a F.S.U of  $\mathbb{K}^+$  and that of  $\mathbb{K}$  is  $\{\varepsilon_1, \sqrt{i\varepsilon_2}, \sqrt{\varepsilon_3}\}$  and  $Q_{\mathbb{K}} = 2$ .*
  - (ii) *Else  $\{\varepsilon_1, \varepsilon_2, \sqrt{\varepsilon_2\varepsilon_3}\}$  is a F.S.U of  $\mathbb{K}^+$  and that of  $\mathbb{K}$  is  $\{\varepsilon_1, \sqrt{\varepsilon_2\varepsilon_3}, \sqrt{i\varepsilon_3}\}$  and  $Q_{\mathbb{K}} = 2$ .*

*Proof.* See Propositions 1, 2, 3 and 4 below. □

*Remark 1.* Our results in this theorem about unit index of  $\mathbb{K}$  are similar to those in Theorem 1 (p. 347) of [12].

**Proposition 1.** *Assume that  $N(\varepsilon_2) = N(\varepsilon_3) = -1$ , then*

- (i) *If  $\varepsilon_1\varepsilon_2\varepsilon_3$  is a square in  $\mathbb{K}^+$ , then  $\{\varepsilon_1, \varepsilon_2, \sqrt{\varepsilon_1\varepsilon_2\varepsilon_3}\}$  is a F.S.U of  $\mathbb{K}^+$  and  $\mathbb{K}$ .*
- (ii) *Else  $\{\varepsilon_1, \varepsilon_2, \varepsilon_3\}$  is a F.S.U of  $\mathbb{K}^+$  and  $\{\varepsilon_1, \varepsilon_2, \sqrt{i\varepsilon_1\varepsilon_2\varepsilon_3}\}$  is a F.S.U of  $\mathbb{K}$ .*

*Proof.* (i) If  $\varepsilon_1\varepsilon_2\varepsilon_3$  is a square in  $\mathbb{K}^+$ , then Lemma 2 (1) yields the result.

(ii) Assume that  $\varepsilon_1\varepsilon_2\varepsilon_3$  is not a square in  $\mathbb{K}^+$ , then from Lemma 2 (2)  $\{\varepsilon_1, \varepsilon_2, \varepsilon_3\}$  is a F.S.U of  $\mathbb{K}^+$ . It remains to determine the F.S.U of  $\mathbb{K}$ .

As  $p_1 \equiv p_2 \equiv 1 \pmod{4}$ , then there exist  $\pi_1, \pi_2, \pi_3$  and  $\pi_4$  in  $\mathbb{Z}[i]$  such that  $p_1 = \pi_1\pi_2$ ,  $p_2 = \pi_3\pi_4$ ,  $\bar{\pi}_1 = \pi_2$  and  $\bar{\pi}_3 = \pi_4$  (the complex conjugate). Let  $\varepsilon_1 = a + b\sqrt{p_1}$ , where  $a, b$  are integers or semi-integers.

(a) Suppose that  $a$  and  $b$  are integers. As  $N(\varepsilon_1) = -1$ , then

$$(a - i)(a + i) = p_1b^2$$

and the gcd of  $a - i$  and  $a + i$  divides 2, so there exist  $b_1$  and  $b_2$  in  $\mathbb{Z}[i]$  such that

$$b = b_1b_2 \text{ and } \begin{cases} a + i = b_1^2\pi_1, \\ a - i = b_2^2\pi_2, \end{cases} \text{ or } \begin{cases} a + i = ib_1^2\pi_1, \\ a - i = -ib_2^2\pi_2; \end{cases}$$

therefore  $2a = b_1^2\pi_1 + b_2^2\pi_2$  or  $2a = ib_1^2\pi_1 - ib_2^2\pi_2$ , hence

$$2\varepsilon_1 = (b_1\sqrt{\pi_1} + b_2\sqrt{\pi_2})^2 \text{ or } 2\varepsilon_1 = (b_1\sqrt{i\pi_1} + b_2\sqrt{-i\pi_2})^2,$$

so

$$\sqrt{2\varepsilon_1} = b_1\sqrt{\pi_1} + b_2\sqrt{\pi_2} \text{ or } 2\sqrt{\varepsilon_1} = b_1(1 + i)\sqrt{\pi_1} + b_2(1 - i)\sqrt{\pi_2},$$

we conclude that

$$\begin{cases} \sqrt{2\pi_1\varepsilon_1} = b_1\pi_1 + b_2\sqrt{p_1} \in \mathbb{K}, \text{ and} \\ \sqrt{2\pi_2\varepsilon_1} = b_1\sqrt{p_1} + b_2\pi_2 \in \mathbb{K}, \text{ or} \\ 2\sqrt{\pi_1\varepsilon_1} = b_1(1 + i)\pi_1 + b_2(1 - i)\sqrt{p_1} \in \mathbb{K} \text{ and} \\ 2\sqrt{\pi_2\varepsilon_1} = b_1(1 + i)\sqrt{p_1} + b_2(1 - i)\pi_2 \in \mathbb{K}. \end{cases} \quad (2)$$

(b) Let  $\varepsilon_1 = \frac{1}{2}(a + b\sqrt{p_1})$ , where  $a, b$  are integers, then

$$(a - 2i)(a + 2i) = \pi_1\pi_2b^2.$$

Proceeding as previously we get the same results.

(c) Let  $\varepsilon_2 = \alpha + \beta\sqrt{2p_2}$ , where  $\alpha, \beta$  are integers, we also find that:

$$\left\{ \begin{array}{l} \sqrt{2(1+i)\pi_3\varepsilon_2} = \beta_1(1+i)\pi_3 + \beta_2\sqrt{2p_2} \in \mathbb{K} \text{ and} \\ \sqrt{2(1-i)\pi_4\varepsilon_2} = \beta_1\sqrt{2p_2} + \beta_2(1-i)\pi_4 \in \mathbb{K} \text{ or} \\ \sqrt{(1+i)\pi_3\varepsilon_2} = \frac{1}{2}(\beta_1(1+i)(1\pm i)\pi_3 + \beta_2(1\mp i)\sqrt{2p_2}) \in \mathbb{K} \text{ and} \\ \sqrt{(1-i)\pi_4\varepsilon_2} = \frac{1}{2}(\beta_1(1\pm i)\sqrt{2p_2} + \beta_2(1-i)(1\mp i)\pi_4) \in \mathbb{K}. \end{array} \right. \quad (3)$$

(d) Applying the same argument to  $\varepsilon_3$ , then we get

$$\left\{ \begin{array}{l} \sqrt{2(1+i)\pi_1\pi_3\varepsilon_3} = y_1(1+i)\pi_1\pi_3 + y_2\sqrt{2p_1p_2} \in \mathbb{K} \text{ and} \\ \sqrt{2(1-i)\pi_2\pi_4\varepsilon_3} = y_1\sqrt{2p_1p_2} + y_2(1-i)\pi_2\pi_4 \in \mathbb{K} \text{ or} \\ \sqrt{(1+i)\pi_1\pi_3\varepsilon_3} = \\ \frac{1}{2}(y_1(1+i)(1\pm i)\pi_1\pi_3 + y_2(1\mp i)\sqrt{2p_1p_2}) \in \mathbb{K} \text{ and} \\ \sqrt{(1-i)\pi_2\pi_4\varepsilon_3} = \\ \frac{1}{2}(y_1(1\pm i)\sqrt{2p_1p_2} + y_2(1\mp i)(1-i)\pi_2\pi_4) \in \mathbb{K}. \end{array} \right. \quad (4)$$

By multiplying the results of equalities (2), (3) and (4), we get

$$\sqrt{\varepsilon_1\varepsilon_2\varepsilon_3} \in \mathbb{K}^+ \text{ or } \sqrt{2\varepsilon_1\varepsilon_2\varepsilon_3} \in \mathbb{K}^+.$$

Finally, note that  $\sqrt{\varepsilon_1\varepsilon_2\varepsilon_3}$  and  $\sqrt{2\varepsilon_1\varepsilon_2\varepsilon_3}$  are not both in  $\mathbb{K}^+$ , since  $\sqrt{2} \in \mathbb{K}^+$ . The rest is a simple deduction from Lemma 2(2).  $\square$

**Proposition 2.** *If  $N(\varepsilon_2) = -N(\varepsilon_3) = 1$ , then the F.S.U of  $\mathbb{K}^+$  is  $\{\varepsilon_1, \varepsilon_2, \varepsilon_3\}$  and that of  $\mathbb{K}$  is  $\{\varepsilon_1, \sqrt{i\varepsilon_2}, \varepsilon_3\}$ .*

*Proof.* As  $N(\varepsilon_2) = 1$ , then, from Lemma 3,  $a \pm 1$  is a square in  $\mathbb{N}$  and  $2\varepsilon_2$  is a square in  $\mathbb{Q}(\sqrt{2p_2})$  ( $\varepsilon_2 = a + b\sqrt{2p_2}$ ). Note that  $\varepsilon_2$  is not a square in  $\mathbb{K}^+$ , else we get  $\sqrt{2} \in \mathbb{K}^+$ , which is false.

Since  $N(\varepsilon_1) = N(\varepsilon_3) = -1$ , then  $\varepsilon_1, \varepsilon_3$  are not squares in  $\mathbb{K}^+$ ; similarly  $\varepsilon_1\varepsilon_2, \varepsilon_1\varepsilon_3, \varepsilon_2\varepsilon_3$  and  $\varepsilon_1\varepsilon_2\varepsilon_3$  are not squares in  $\mathbb{K}^+$ , else we will find that  $i \in \mathbb{K}^+$ , which is absurd. Therefore  $\{\varepsilon_1, \varepsilon_2, \varepsilon_3\}$  is the F.S.U of  $\mathbb{K}^+$  and as  $2\varepsilon_2$  is a square in  $\mathbb{K}^+$ , then, from Lemma 1,  $\{\varepsilon_1, \sqrt{i\varepsilon_2}, \varepsilon_3\}$  is the F.S.U of  $\mathbb{K}$ , which implies that  $Q_{\mathbb{K}} = 2$ .  $\square$

**Proposition 3.** *Assume that  $N(\varepsilon_3) = -N(\varepsilon_2) = 1$ , then*

- (i) *If  $2p_1(x \pm 1)$  is a square in  $\mathbb{N}$ , then  $\{\varepsilon_1, \varepsilon_2, \sqrt{\varepsilon_3}\}$  is a F.S.U of  $\mathbb{K}^+$  and  $\mathbb{K}$ .*
- (ii) *Else  $\{\varepsilon_1, \varepsilon_2, \varepsilon_3\}$  is a F.S.U of  $\mathbb{K}^+$  and that of  $\mathbb{K}$  is  $\{\varepsilon_1, \varepsilon_2, \sqrt{i\varepsilon_3}\}$ .*

*Proof.* Since  $N(\varepsilon_1) = N(\varepsilon_2) = -1$  and  $N(\varepsilon_3) = 1$ , then only  $\varepsilon_3$  can be a square in  $\mathbb{K}^+$ . The equality  $N(\varepsilon_3) = 1$  implies

$$(x-1)(x+1) = 2p_1p_2y^2,$$

and since  $2p_1p_2 \equiv 2 \pmod{4}$ , thus  $2 \nmid x$ ,  $2 \mid y$  and  $(x+1, x-1) = 2$ , hence with  $A = (x \pm 1)/2$ ,  $B = (x \mp 1)/2$  and  $y = 2z$  we get

$$AB = 2p_1p_2z^2,$$

without loss of generality we may assume  $2 \mid A$ . Thus according to Lemma 4 and to the decomposition uniqueness in  $\mathbb{Z}$ , we get three possibilities:

Case 1: If  $p_1 \nmid A$ ,  $p_2 \nmid A$ , then there exist  $z_1, z_2$  in  $\mathbb{N}$  with  $z = z_1z_2$  such that

$$A = 2z_1^2, \quad B = p_1p_2z_2^2,$$

hence

$$x \pm 1 = (2z_1)^2, \quad x \mp 1 = 2p_1p_2z_2^2.$$

Therefore  $\sqrt{2\varepsilon_3} = \frac{1}{2}(2z_1 + z_2\sqrt{2p_1p_2}) \in \mathbb{Q}(\sqrt{2p_1p_2})$  and  $\sqrt{\varepsilon_3} \notin \mathbb{K}^+$ .

Case 2: If  $p_1 \mid A$ ,  $p_2 \nmid A$ , then we get

$$A = 2p_1z_1^2, \quad B = p_2z_2^2,$$

hence

$$x \pm 1 = p_1(2z_1)^2, \quad x \mp 1 = 2p_2z_2^2.$$

Therefore  $\sqrt{2\varepsilon_3} = 2z_1\sqrt{p_1} + z_2\sqrt{2p_2} \in \mathbb{K}^+$  and  $\sqrt{\varepsilon_3} \notin \mathbb{K}^+$ .

Case 3: If  $p_1 \nmid A$ ,  $p_2 \mid A$ , then we get

$$A = 2p_2z_1^2, \quad B = p_1z_2^2,$$

hence

$$x \pm 1 = p_2(2z_1)^2, \quad x \mp 1 = 2p_1z_2^2.$$

Therefore  $\sqrt{\varepsilon_3} = z_1\sqrt{2p_2} + z_2\sqrt{p_1} \in \mathbb{K}^+$ .

As a result if  $2p_1(x \mp 1)$  is a square in  $\mathbb{N}$ , then  $\{\varepsilon_1, \varepsilon_2, \sqrt{\varepsilon_3}\}$  is a *F.S.U* of  $\mathbb{K}^+$ ,  $\mathbb{K}$  and thus  $Q_{\mathbb{K}} = 1$ ; else  $\sqrt{\varepsilon_3} \notin \mathbb{K}^+$  and  $\sqrt{2\varepsilon_3} \in \mathbb{K}^+$ , this yields that  $\{\varepsilon_1, \varepsilon_2, \varepsilon_3\}$  is a *F.S.U* of  $\mathbb{K}^+$  and from Lemma 1,  $\{\varepsilon_1, \varepsilon_2, \sqrt{i\varepsilon_3}\}$  is a *F.S.U* of  $\mathbb{K}$ , so  $Q_{\mathbb{K}} = 2$ .  $\square$

**Proposition 4.** Assume that  $N(\varepsilon_3) = N(\varepsilon_2) = 1$ , then

- (i) If  $2p_1(x \pm 1)$  is a square in  $\mathbb{N}$ , then  $\{\varepsilon_1, \varepsilon_2, \sqrt{\varepsilon_3}\}$  is a *F.S.U* of  $\mathbb{K}^+$  and that of  $\mathbb{K}$  is  $\{\varepsilon_1, \sqrt{i\varepsilon_2}, \sqrt{\varepsilon_3}\}$ .
- (ii) Else  $\{\varepsilon_1, \varepsilon_2, \sqrt{\varepsilon_2\varepsilon_3}\}$  is a *F.S.U* of  $\mathbb{K}^+$  and that of  $\mathbb{K}$  is  $\{\varepsilon_1, \sqrt{\varepsilon_2\varepsilon_3}, \sqrt{i\varepsilon_3}\}$ .

*Proof.* Since  $N(\varepsilon_2) = 1$ , then, from Lemma 3,  $2\varepsilon_2$  is a square in  $k_2$ ; hence  $\varepsilon_2$  is not a square in  $\mathbb{K}^+$ .

Proceeding as in Proposition 3 we get three cases.

(a) If  $x \pm 1$  is a square in  $\mathbb{N}$ , then  $\sqrt{\varepsilon_3} \notin \mathbb{K}^+$  and  $\sqrt{2\varepsilon_3} \in \mathbb{K}^+$ , so  $\sqrt{\varepsilon_2\varepsilon_3} \in \mathbb{K}^+$ ; hence  $\{\varepsilon_1, \varepsilon_2, \sqrt{\varepsilon_2\varepsilon_3}\}$  is a *F.S.U* of  $\mathbb{K}^+$  and by Lemma 1,  $\{\varepsilon_1, \sqrt{\varepsilon_2\varepsilon_3}, \sqrt{i\varepsilon_3}\}$  is a *F.S.U* of  $\mathbb{K}$ , thus  $Q_{\mathbb{K}} = 2$ . It should be noted that, by Lemma 1, we can take as a *F.S.U* of  $\mathbb{K}$  one of the following systems  $\{\varepsilon_1, \sqrt{\varepsilon_2\varepsilon_3}, \sqrt{i\varepsilon_2}\}$ ,  $\{\varepsilon_1, \sqrt{i\varepsilon_2}, \sqrt{i\varepsilon_3}\}$ .

(b) If  $2p_1(x \pm 1)$  is a square in  $\mathbb{N}$ , then  $\sqrt{\varepsilon_3} \in \mathbb{K}^+$ , so  $\sqrt{2\varepsilon_2\varepsilon_3} \in \mathbb{K}^+$ . Thus  $\{\varepsilon_1, \varepsilon_2, \sqrt{\varepsilon_3}\}$  is a *F.S.U* of  $\mathbb{K}^+$  and by Lemma 1,  $\{\varepsilon_1, \sqrt{i\varepsilon_2}, \sqrt{\varepsilon_3}\}$  is a *F.S.U* of  $\mathbb{K}$ , therefore  $Q_{\mathbb{K}} = 2$ .

(c) If  $p_1(x \pm 1)$  is a square in  $\mathbb{N}$ , then  $\sqrt{\varepsilon_3} \notin \mathbb{K}^+$  and  $\sqrt{2\varepsilon_3} \in \mathbb{K}^+$ , so  $\sqrt{\varepsilon_2\varepsilon_3} \in \mathbb{K}^+$ . The rest is as the case (a).  $\square$

**2.2. F.S.U OF THE FIELD  $\mathbb{K} = \mathbb{Q}(\sqrt{2}, \sqrt{p_1p_2}, i)$ .** Let  $\mathbb{K} = \mathbb{k}(\sqrt{2}) = \mathbb{Q}(\sqrt{2}, \sqrt{p_1p_2}, i)$ . Denote by  $\varepsilon_1$  (resp.  $\varepsilon_2, \varepsilon_3$ ) the fundamental unit of  $\mathbb{Q}(\sqrt{2})$  (resp.  $\mathbb{Q}(\sqrt{p_1p_2})$ ,  $\mathbb{Q}(\sqrt{2p_1p_2})$ ). Put  $I = \{0, 1\}$  and  $\varepsilon_3 = x + y\sqrt{2p_1p_2}$ . Our aim in this subsection is to state the following theorem, but first let us show the lemma.

**Lemma 7.** *Put  $\varepsilon_2 = a + b\sqrt{p_1p_2}$ ; if  $N(\varepsilon_2) = 1$ , then  $a \pm 1$  is not a square in  $\mathbb{N}$ .*

*Proof.* As  $p_1p_2 \equiv 1 \pmod{4}$ , then, from Lemma 6, the unit index of  $\mathbb{Q}(\sqrt{p_1p_2}, i)$  is equal to 1; since  $N(\varepsilon_2) = 1$ , so assertion 3.(1).(ii) on p.19 of [1], yields that  $a \pm 1$  is not a square in  $\mathbb{N}$ .  $\square$

**Theorem 2.** *Keep notations mentioned above, Then  $Q_{\mathbb{K}} = 1$  and*

- (1) *If  $N(\varepsilon_2) = N(\varepsilon_3) = -1$ , then*
  - (i) *If  $\varepsilon_1\varepsilon_2\varepsilon_3$  is a square in  $\mathbb{K}^+$ , then  $\{\varepsilon_1, \varepsilon_2, \sqrt{\varepsilon_1\varepsilon_2\varepsilon_3}\}$  is a *F.S.U* of  $\mathbb{K}^+, \mathbb{K}$ .*
  - (ii) *Else  $\{\varepsilon_1, \varepsilon_2, \varepsilon_3\}$  is a *F.S.U* of  $\mathbb{K}^+, \mathbb{K}$ .*
- (2) *If  $N(\varepsilon_2) = -N(\varepsilon_3) = 1$ , then the *F.S.U* of  $\mathbb{K}^+, \mathbb{K}$  is  $\{\varepsilon_1, \varepsilon_2, \varepsilon_3\}$ .*
- (3) *If  $N(\varepsilon_3) = -N(\varepsilon_2) = 1$ , then*
  - (i) *If  $x \pm 1$  is a square in  $\mathbb{N}$ , then  $\{\varepsilon_1, \varepsilon_2, \sqrt{\varepsilon_3}\}$  is a *F.S.U* of  $\mathbb{K}^+, \mathbb{K}$ .*
  - (ii) *Else  $\{\varepsilon_1, \varepsilon_2, \varepsilon_3\}$  is a *F.S.U* of  $\mathbb{K}^+, \mathbb{K}$ .*
- (4) *If  $N(\varepsilon_3) = N(\varepsilon_2) = 1$ , then*
  - (i) *If  $x \pm 1$  is a square in  $\mathbb{N}$ , then  $\{\varepsilon_1, \varepsilon_2, \sqrt{\varepsilon_3}\}$  is a *F.S.U* of  $\mathbb{K}^+, \mathbb{K}$ .*
  - (ii) *Else  $\{\varepsilon_1, \varepsilon_2, \sqrt{\varepsilon_2\varepsilon_3}\}$  is a *F.S.U* of  $\mathbb{K}^+, \mathbb{K}$ .*



*Proof.* See Propositions 5, 6, 7 and 8 below.  $\square$

*Remark 2.* The unit index of  $\mathbb{K}$  is always equal to 1, which is compatible with theorem 2 of [12].

**Proposition 5.** *Assume that  $N(\varepsilon_2) = -N(\varepsilon_3) = 1$ , then  $\{\varepsilon_1, \varepsilon_2, \varepsilon_3\}$  is a F.S.U of  $\mathbb{K}^+$  and  $\mathbb{K}$ .*

*Proof.* Since  $\varepsilon_1, \varepsilon_3$  have negative norms, then they are not squares in  $\mathbb{K}^+$ , similarly  $\varepsilon_1\varepsilon_2, \varepsilon_1\varepsilon_3, \varepsilon_2\varepsilon_3$  and  $\varepsilon_1\varepsilon_2\varepsilon_3$  are not squares in  $\mathbb{K}^+$ , else by taking a suitable norm we get  $i \in \mathbb{K}^+$ , which is false. Furthermore  $(2 + \sqrt{2})\varepsilon_1^i\varepsilon_2^j\varepsilon_3^k$  cannot be a square in  $\mathbb{K}^+$ , for all  $i, j$  and  $k$  of  $\mathbb{I}$ , as otherwise with some  $\alpha \in \mathbb{K}^+$  we would have  $\alpha^2 = (2 + \sqrt{2})\varepsilon_1^i\varepsilon_2^j\varepsilon_3^k$ , so  $N_{\mathbb{K}^+/\mathbb{K}_2}^2(\alpha) = 2(-1)^{i+k}\varepsilon_2^{2j}$ , yielding that  $\sqrt{2} \in \mathbb{Q}(\sqrt{p_1p_2})$ , which is absurd.

If  $\varepsilon_2 = a + b\sqrt{p_1p_2}$ , then  $a^2 - 1 = b^2p_1p_2$ . Proceeding as in Proposition 3 and taking into account Lemma 4, we get the following cases.

(i) If  $p_1p_2(a \pm 1)$  is a square in  $\mathbb{N}$ , then there exists  $(b_1, b_2) \in \mathbb{Z}^2$  such that

$$a \mp 1 = b_1^2 \text{ and } a \pm 1 = b_2^2p_1p_2,$$

so  $a \mp 1$  is a square in  $\mathbb{N}$ , which contradicts Lemma 7.

(ii) If  $p_1(a \pm 1)$  is a square in  $\mathbb{N}$ , then there exists  $(b_1, b_2) \in \mathbb{Z}^2$  such that

$$a \pm 1 = p_1b_1^2 \text{ and } a \mp 1 = p_2b_2^2,$$

thus  $\sqrt{\varepsilon_2} = \frac{1}{2}(b_1\sqrt{2p_1} + b_2\sqrt{2p_2})$ , so  $\sqrt{\varepsilon_2} \notin \mathbb{K}^+$ ,  $\sqrt{p_1\varepsilon_2} \in \mathbb{K}^+$  and  $\sqrt{p_2\varepsilon_2} \in \mathbb{K}^+$ .

(iii) If  $2p_1(a \pm 1)$  is a square in  $\mathbb{N}$ , then the same argument shows

that  $\sqrt{\varepsilon_2} = b_1\sqrt{p_1} + b_2\sqrt{p_2}$ ,  $\sqrt{p_1\varepsilon_2} \in \mathbb{K}^+$  and  $\sqrt{p_2\varepsilon_2} \in \mathbb{K}^+$ .

Therefore we deduce that  $\{\varepsilon_1, \varepsilon_2, \varepsilon_3\}$  is a F.S.U of  $\mathbb{K}^+$  and from Lemma 1,  $\mathbb{K}$  has the same F.S.U.  $\square$

**Proposition 6.** *Assume that  $N(\varepsilon_2) = N(\varepsilon_3) = -1$ , then*

(i) *If  $\varepsilon_1\varepsilon_2\varepsilon_3$  is a square in  $\mathbb{K}_3^+$ , then  $\{\varepsilon_1, \varepsilon_2, \sqrt{\varepsilon_1\varepsilon_2\varepsilon_3}\}$  is a F.S.U of  $\mathbb{K}_3^+$  and  $\mathbb{K}_3$ .*

(ii) *Else  $\{\varepsilon_1, \varepsilon_2, \varepsilon_3\}$  is a F.S.U of  $\mathbb{K}_3^+$  and  $\mathbb{K}_3$ .*

*Proof.* We proceed as in Proposition 5 to prove that  $(2 + \sqrt{2})\varepsilon_1^i\varepsilon_2^j\varepsilon_3^k$  is not a square in  $\mathbb{K}^+$ , for all  $i, j$  and  $k$  of  $\mathbb{I}$ , and we apply Lemmas 2, 1 and Remark 3 bellow.  $\square$

*Remark 3.* We proceed as in the proof of Proposition 1 to prove the following:

(a) If  $\varepsilon_2 = x + y\sqrt{p_1p_2}$ , then

$$\begin{cases} \sqrt{\varepsilon_2} = y_1\sqrt{\pi_1\pi_3} + y_2\sqrt{\pi_2\pi_4}, & \text{or} \\ \sqrt{\varepsilon_2} = y_1\sqrt{\pi_1\pi_4} + y_2\sqrt{\pi_2\pi_3}, & \text{or} \\ \sqrt{2\varepsilon_2} = y_1\sqrt{\pi_1\pi_3} + y_2\sqrt{\pi_2\pi_4}, & \text{or} \\ \sqrt{2\varepsilon_2} = y_1\sqrt{\pi_1\pi_4} + y_2\sqrt{\pi_2\pi_3}, \end{cases} \quad (5)$$

where  $y_i$  are in  $\mathbb{Z}[i]$  or  $\frac{1}{2}\mathbb{Z}[i]$ .

(b) If  $\varepsilon_3 = a + b\sqrt{2p_1p_2}$ , then

$$\begin{cases} \sqrt{\varepsilon_3} = b_1\sqrt{(1+i)\pi_1\pi_3} + b_2\sqrt{(1-i)\pi_2\pi_4}, & \text{or} \\ \sqrt{\varepsilon_3} = b_1\sqrt{(1+i)\pi_1\pi_4} + b_2\sqrt{(1-i)\pi_2\pi_3}, & \text{or} \\ \sqrt{2\varepsilon_3} = b_1\sqrt{(1+i)\pi_1\pi_3} + b_2\sqrt{(1-i)\pi_2\pi_4}, & \text{or} \\ \sqrt{2\varepsilon_3} = b_1\sqrt{(1+i)\pi_1\pi_4} + b_2\sqrt{(1-i)\pi_2\pi_3}, \end{cases} \quad (6)$$

where  $b_i$  are in  $\mathbb{Z}[i]$  or  $\frac{1}{2}\mathbb{Z}[i]$ .

Note at the end that:

$$\sqrt{2\varepsilon_1} = \sqrt{1+i} + \sqrt{1-i}. \quad (7)$$

So by multiplying results of equalities (5), (6) and (7) we get

$$\begin{aligned} \sqrt{\varepsilon_1\varepsilon_2\varepsilon_3} &= \alpha + \beta\sqrt{2} + \gamma\sqrt{p_1p_2} + \delta\sqrt{2p_1p_2} \in \mathbb{Q}(\sqrt{2}, \sqrt{p_1p_2}) \text{ or} \\ \sqrt{\varepsilon_1\varepsilon_2\varepsilon_3} &= \alpha\sqrt{p_1} + \beta\sqrt{p_2} + \gamma\sqrt{2p_1} + \delta\sqrt{2p_2} \notin \mathbb{Q}(\sqrt{2}, \sqrt{p_1p_2}), \end{aligned}$$

where  $\alpha, \beta, \gamma$  and  $\delta$  are in  $\mathbb{Q}$ .

**Proposition 7.** Assume that  $N(\varepsilon_3) = -N(\varepsilon_2) = 1$ , then

- (i) If  $x \pm 1$  is a square in  $\mathbb{N}$ , then  $\{\varepsilon_1, \varepsilon_2, \sqrt{\varepsilon_3}\}$  is a F.S.U of  $\mathbb{K}$  and  $\mathbb{K}^+$ .
- (ii) Else  $\{\varepsilon_1, \varepsilon_2, \varepsilon_3\}$  is a F.S.U of  $\mathbb{K}^+$  and  $\mathbb{K}$ .

*Proof.* As the norms of  $\varepsilon_1, \varepsilon_2$  are negative, then proceeding as in Proposition 5, we prove that only  $\varepsilon_3$  can be a square in  $\mathbb{K}^+$ .

(i) According to Lemma 5,  $2\varepsilon_3$  is a square in  $\mathbb{Q}(\sqrt{2p_1p_2})$ , and since  $\sqrt{2} \in \mathbb{K}^+$ , so  $\sqrt{\varepsilon_3} \in \mathbb{K}^+$ , which yields that  $\{\varepsilon_1, \varepsilon_2, \sqrt{\varepsilon_3}\}$  is a F.S.U of  $\mathbb{K}$  and, by Lemma 1, is also a F.S.U of  $\mathbb{K}^+$ .

(ii)  $\varepsilon_3$  is not a square in  $\mathbb{K}^+$ , then  $\{\varepsilon_1, \varepsilon_2, \varepsilon_3\}$  is a F.S.U of  $\mathbb{K}, \mathbb{K}^+$  (Lemma 1). □

**Proposition 8.** Assume that  $N(\varepsilon_3) = N(\varepsilon_2) = 1$ , then

- (i) If  $x \pm 1$  a square in  $\mathbb{N}$ , then  $\{\varepsilon_1, \varepsilon_2, \sqrt{\varepsilon_3}\}$  is a F.S.U of  $\mathbb{K}$  and  $\mathbb{K}^+$ .
- (ii) Else  $\{\varepsilon_1, \varepsilon_2, \sqrt{\varepsilon_2\varepsilon_3}\}$  is a F.S.U of  $\mathbb{K}^+$  and  $\mathbb{K}$ .

*Proof.* Since  $N(\varepsilon_1) = -1$ , then  $\varepsilon_1, \varepsilon_1\varepsilon_2, \varepsilon_1\varepsilon_3, \varepsilon_1\varepsilon_2\varepsilon_3$  and  $(2 + \sqrt{2})\varepsilon_1^i\varepsilon_2^j\varepsilon_3^k$  are not squares in  $\mathbb{K}^+$ , for all  $i, j$  and  $k$  in  $\mathbb{I}$ .

As  $N(\varepsilon_2) = 1$ , then from Lemma 7, there exist  $y_1, y_2$  in  $\mathbb{Z}$  such that:

$$\sqrt{2\varepsilon_2} = y_1\sqrt{p_1} + y_2\sqrt{p_2}. \quad (8)$$

The equality  $N(\varepsilon_3) = 1$  and the Lemma 4 imply the existence of  $a_1, a_2, n$  and  $m$  in  $\mathbb{N}$ , such that

$$n = p_1 \text{ and } m = 2p_2 \text{ or } n = 2p_1 \text{ and } m = p_2, \text{ and } \begin{cases} x \pm 1 = a_1^2, \\ x \mp 1 = 2p_1p_2, \end{cases} \quad \text{or} \quad \begin{cases} x \pm 1 = na_1^2, \\ x \mp 1 = ma_2^2; \end{cases}$$

we get

$$\sqrt{\varepsilon_3} = \frac{1}{2}(a_1\sqrt{2} + 2a_2\sqrt{p_1p_2}) \text{ or } \sqrt{\varepsilon_3} = \frac{1}{2}(2a_1\sqrt{n} + a_2\sqrt{m}). \quad (9)$$

From equalities (8) and (9) we deduce that if  $x \pm 1$  is a square in  $\mathbb{N}$ , then  $\sqrt{\varepsilon_3} \in \mathbb{K}^+$ , else  $\sqrt{\varepsilon_2\varepsilon_3} \in \mathbb{K}^+$ . Hence the results.  $\square$

### 3. Some results

In what remains of this paper, we adopt the following notations. Let  $p_1 = e^2 + 4f^2 = \pi_1\pi_2$ ,  $p_2 = g^2 + 4h^2 = \pi_3\pi_4$ ,  $\pi_1 = e + 2if$ ,  $\pi_2 = e - 2if$ ,  $\pi_3 = g + 2ih$ ,  $\pi_4 = g - 2ih$ . Let  $\mathcal{H}_j$  be the prime ideal of  $\mathbb{k}$  above  $\pi_j$ , hence  $\mathcal{H}_j^2 = (\pi_j)$ , for all  $j \in \{1, 2, 3, 4\}$ . As 2 is totally ramified in  $\mathbb{k}$ , let  $\mathcal{H}_0$  be the prime ideal of  $\mathbb{k}$  above  $1 + i$ , so  $\mathcal{H}_0^2 = (1 + i)$

**Proposition 9.** *Let  $d$  be a square-free integer,  $k = \mathbb{Q}(\sqrt{d}, i)$ ,  $a + ib$  an element of  $\mathbb{Z}[i]$  and  $\mathcal{H}$  an ideal of  $k$  such that  $\mathcal{H}^2 = (a + ib)$ . Put  $\varepsilon_d = x + y\sqrt{d}$  the fundamental unit of  $\mathbb{Q}(\sqrt{d})$ , so*

(1) *If  $\sqrt{a^2 + b^2} \notin \mathbb{Q}(\sqrt{d})$ , then  $\mathcal{H}$  is not principal in  $k$ .*

(2) *If  $a^2 + b^2 = d$ , then*

(a) *If  $N(\varepsilon_d) = 1$ , then  $\mathcal{H}$  is not principal in  $k$ .*

(b) *If  $N(\varepsilon_d) = -1$ , then:*

(i) *If  $(ax \pm yd) \pm b$  or  $2(-xb \pm yd) \pm a$  is a square in  $\mathbb{N}$ , then  $\mathcal{H}$  is principal in  $k$ .*

(ii) *Else  $\mathcal{H}$  is not principal in  $k$ .*

*Proof.* See Proposition 1 of [6]  $\square$

**Proposition 10.** *Let  $d$  be a composite integer, even, square-free and product at least of three primes. Let  $k = \mathbb{Q}(\sqrt{d}, i)$ ,  $p$  an odd prime and  $\mathcal{H}$  an ideal of  $k$  such that  $\mathcal{H}^2 = (p)$ . Let  $\varepsilon_d = x + y\sqrt{d}$  denote the fundamental unit of  $\mathbb{Q}(\sqrt{d})$ . Then*

- (1) *If  $N(\varepsilon_d) = -1$ , then  $\mathcal{H}$  is not principal in  $k$ .*
- (2) *If  $N(\varepsilon_d) = 1$ , then*
  - (i) *If  $\{\varepsilon_d\}$  is F.S.U of  $k$ , then  $\mathcal{H}$  is principal in  $k$  if and only if  $2p(x \pm 1)$  or  $p(x \pm 1)$  is a square in  $\mathbb{N}$ .*
  - (ii) *Else  $\mathcal{H}$  is not principal in  $k$ .*

*Proof.* See Proposition 2 of [6]  $\square$

We finish this paragraph by the following two Lemmas.

**Lemma 8.** *Let  $d = 2p_1p_2$  and  $\varepsilon_d = x + y\sqrt{d}$ . If  $N(\varepsilon_d) = -1$ , then there exist  $y_1$  and  $y_2$  in  $\mathbb{Z}[i]$  such that  $\sqrt{\varepsilon_d}$  takes one of following forms:*

- (1)  $\frac{1}{2}[y_1(1+i)\sqrt{(1+i)\pi_1\pi_3} + y_2(1-i)\sqrt{(1-i)\pi_2\pi_4}]$ .
- (2)  $\frac{1}{2}[y_1(1+i)\sqrt{(1+i)\pi_1\pi_4} + y_2(1-i)\sqrt{(1-i)\pi_2\pi_3}]$ .

*Proof.* As  $N(\varepsilon_d) = -1$ , then  $x^2 + 1 = y^2d$ , hence there exist  $y_1, y_2$  in  $\mathbb{Z}[i]$  such that

$$\begin{cases} x \pm i = (1+i)\pi_1\pi_3y_1^2, \\ x \mp i = (1-i)\pi_2\pi_4y_2^2, \end{cases} \text{ or } \begin{cases} x \pm i = (1+i)\pi_1\pi_4y_1^2, \\ x \mp i = (1-i)\pi_2\pi_3y_2^2, \end{cases} \text{ or } \begin{cases} x \pm i = i(1+i)\pi_1\pi_3y_1^2, \\ x \mp i = -i(1-i)\pi_2\pi_4y_2^2, \end{cases} \text{ or } \begin{cases} x \pm i = i(1+i)\pi_1\pi_4y_1^2, \\ x \mp i = -i(1-i)\pi_2\pi_3y_2^2; \end{cases}$$

therefore

$$2x = (1+i)\pi_1\pi_3y_1^2 + (1-i)\pi_2\pi_4y_2^2 \text{ or } 2x = (1+i)\pi_1\pi_4y_1^2 + (1-i)\pi_2\pi_3y_2^2 \text{ or } 2x = i(1+i)\pi_1\pi_3y_1^2 - i(1-i)\pi_2\pi_4y_2^2 \text{ or } 2x = i(1+i)\pi_1\pi_4y_1^2 - i(1-i)\pi_2\pi_3y_2^2,$$

so

$$\begin{aligned} \varepsilon_d &= \frac{1}{2}[y_1\sqrt{(1+i)\pi_1\pi_3} + y_2\sqrt{(1-i)\pi_2\pi_4}]^2 \text{ or} \\ \varepsilon_d &= \frac{1}{2}[y_1\sqrt{(1+i)\pi_1\pi_4} + y_2\sqrt{(1-i)\pi_2\pi_3}]^2 \text{ or} \\ \varepsilon_d &= \frac{1}{2}[y_1(1+i)\sqrt{\frac{(1+i)\pi_1\pi_3}{2}} + y_2(1-i)\sqrt{\frac{(1-i)\pi_2\pi_4}{2}}]^2 \text{ or} \\ \varepsilon_d &= \frac{1}{2}[y_1(1+i)\sqrt{\frac{(1+i)\pi_1\pi_4}{2}} + y_2(1-i)\sqrt{\frac{(1-i)\pi_2\pi_3}{2}}]^2, \end{aligned}$$

hence

$$\begin{aligned}\sqrt{\varepsilon_d} &= \frac{1}{2}[y_1(1+i)\sqrt{(1\pm i)\pi_1\pi_3} + y_2(1-i)\sqrt{(1\mp i)\pi_2\pi_4}] \text{ or} \\ \sqrt{\varepsilon_d} &= \frac{1}{2}[y_1(1+i)\sqrt{(1\pm i)\pi_1\pi_4} + y_2(1-i)\sqrt{(1\mp i)\pi_2\pi_3}].\end{aligned}$$

□

**Lemma 9.** *Let  $d = 2p_1p_2$  and  $\mathbb{k} = \mathbb{Q}(\sqrt{d}, i)$ . If  $N(\varepsilon_d) = -1$ , then*

- (i) *If  $\sqrt{\varepsilon_d}$  takes the form (1) of Lemma 8, then  $\mathcal{H}_0\mathcal{H}_1\mathcal{H}_3$  and  $\mathcal{H}_0\mathcal{H}_2\mathcal{H}_4$  are principal in  $\mathbb{k}$ .*
- (ii) *If  $\sqrt{\varepsilon_d}$  takes the form (2) of Lemma 8, then  $\mathcal{H}_0\mathcal{H}_1\mathcal{H}_4$  and  $\mathcal{H}_0\mathcal{H}_2\mathcal{H}_3$  are principal in  $\mathbb{k}$ .*

*Proof.* It is easy to see that for all  $j \in \{1, 2, 3, 4\}$ ,  $\pi_j$  is ramified in  $\mathbb{k}/\mathbb{Q}(i)$ , thus  $\mathcal{H}_j^2 = (\pi_j)$ . On the other hand, 2 is totally ramified in  $\mathbb{k}$  and  $\mathcal{H}_0^2 = (1+i)$ . Hence for all  $j \in \{0, 1, 2, 3, 4\}$ , Proposition 9, states that  $\mathcal{H}_j$  is not principal in  $\mathbb{k}$ , since  $\sqrt{p_1}$ ,  $\sqrt{p_2}$  and  $\sqrt{2}$  are not in  $\mathbb{Q}(\sqrt{2p_1p_2})$ .

The ideal  $\mathcal{H}_0\mathcal{H}_1\mathcal{H}_3$  is principal in  $\mathbb{k}$ , if and only if there exists a unit  $\varepsilon \in \mathbb{k}$  such that

$$(1+i)\pi_1\pi_3\varepsilon = \alpha^2, \quad (10)$$

where  $\alpha \in \mathbb{k}$ . As  $N(\varepsilon_d) = -1$ , so, by Lemma 5,  $Q_{\mathbb{k}} = 1$ ; hence  $\varepsilon$  is either real or purely imaginary.

Put  $\alpha = \alpha_1 + i\alpha_2$ , with  $\alpha_1, \alpha_2 \in \mathbb{Q}(\sqrt{2p_1p_2})$ , and suppose  $\varepsilon$  is real (same proof if it is purely imaginary), since  $\pi_1\pi_3 = (e+2if)(g+2ih) = (eg-4fh) + 2i(eh+gf)$ , then the equation (10) is equivalent to

$$\alpha_1^2 - \alpha_2^2 + 2i\alpha_1\alpha_2 = \varepsilon[(eg-4fh) - 2(eh+fg)] + i\varepsilon_d[(eg-4fh) + 2(eh+gf)],$$

hence

$$\begin{aligned}\alpha_1^2 - \alpha_2^2 &= \varepsilon[(eg-4fh) - 2(eh+fg)], \text{ and} \\ 2\alpha_1\alpha_2 &= \varepsilon[(eg-4fh) + 2(eh+gf)],\end{aligned}$$

so we get

$$\alpha_2 = \frac{\varepsilon[(eg-4fh) + 2(eh+gf)]}{2\alpha_1},$$

thus

$$4\alpha_1^4 - 4\varepsilon[(eg-4fh) - 2(eh+fg)]\alpha_1^2 - [(eg-4fh) + 2(eh+gf)]^2\varepsilon^2 = 0,$$

the discriminant of this equation is  $\Delta' = 4\varepsilon^2d$ , which implies that

$$\alpha_1^2 = \frac{\varepsilon}{4}[2[(eg-4fh) - 2(eh+fg)] \pm 2\sqrt{d}].$$

On the other hand,

$$(1+i)\pi_1\pi_3 + (1-i)\pi_2\pi_4 = 2(eg - 4fh) - 4(eh + fg) \text{ and } \\ \sqrt{d} = \sqrt{(1-i)\pi_1\pi_3}\sqrt{(1+i)\pi_2\pi_4},$$

then

$$\alpha_1^2 = \frac{\varepsilon}{4}(\sqrt{(1-i)\pi_1\pi_3} + \sqrt{(1+i)\pi_2\pi_4})^2, \text{ so } \\ \alpha_1 = \frac{\sqrt{\varepsilon}}{2}(\sqrt{(1-i)\pi_1\pi_3} + \sqrt{(1+i)\pi_2\pi_4}),$$

therefore if  $\varepsilon = \varepsilon_d$  and  $\sqrt{\varepsilon_d}$  takes the value (1) of Lemma 8, we get

$$\alpha_1 = \frac{1}{4}(2y_1\pi_1\pi_3 + 2y_2\pi_2\pi_4 + (y_1(1+i) + y_2(1-i))\sqrt{d}),$$

and

$$\alpha_2 = \frac{\varepsilon_d[(eg - 4fh) + 2(eh + gf)]}{2\alpha_1},$$

and it is easy to see that  $\alpha_1, \alpha_2 \in \mathbb{Q}(\sqrt{2p_1p_2})$ ; hence  $\mathcal{H}_0\mathcal{H}_1\mathcal{H}_3$  is principal in  $\mathbb{k}$ . By writing  $\pi_2\pi_4$  in terms of e, f, g and h we prove similarly that  $\mathcal{H}_0\mathcal{H}_2\mathcal{H}_4$  is principal.

Proceeding similarly we prove that  $\mathcal{H}_0\mathcal{H}_1\mathcal{H}_4$  and  $\mathcal{H}_0\mathcal{H}_2\mathcal{H}_3$  are principal in  $\mathbb{k}$ , if  $\sqrt{\varepsilon_d}$  takes the value (2) of Lemma 8.  $\square$

#### 4. The strongly ambiguous classes of $\mathbb{k}/\mathbb{Q}(i)$

Let  $F = \mathbb{Q}(i)$  and  $\text{Galois}(\mathbb{k}/F) = \langle \sigma \rangle$ . We denote by  $Am(\mathbb{k}/F)$  the group of the ambiguous classes of  $\mathbb{k}/F$ , that are classes of  $\mathbb{k}$  fixed under  $\sigma$ , we denote also by  $Am_s(\mathbb{k}/F)$  the subgroup of  $Am(\mathbb{k}/F)$  generated by the strongly ambiguous classes, which are classes of  $\mathbb{k}$  containing at least one ideal invariant under  $\sigma$ . The genus number,  $[(\mathbb{k}/F)^* : \mathbb{k}]$ , is given by the ambiguous class number formula (see [7]):

$$|Am(\mathbb{k}/F)| = [(\mathbb{k}/F)^* : \mathbb{k}] = \frac{h(F)2^{t-1}}{[E_F : E_F \cap N_{\mathbb{k}/F}(\mathbb{k}^\times)]}, \quad (11)$$

where  $h(F)$  is the class number of  $F$ ,  $t$  is the number of finite and infinite primes of  $F$  ramified in  $\mathbb{k}/F$ . Moreover as the class number of  $F$  is equal to 1, so it is well known that

$$|Am(\mathbb{k}/F)| = [(\mathbb{k}/F)^* : \mathbb{k}] = 2^r, \quad (12)$$

where  $r = \text{rank } \mathbf{C}_{\mathbb{k},2}$ . The relation between  $|Am(\mathbb{k}/F)|$  and  $|Am_s(\mathbb{k}/F)|$  is given by the formula:

$$\frac{|Am(\mathbb{k}/F)|}{|Am_s(\mathbb{k}/F)|} = [E_F \cap N_{\mathbb{k}/F}(\mathbb{k}^\times) : N_{\mathbb{k}/F}(E_{\mathbb{k}})]. \quad (13)$$

Since  $\mathcal{H}_0^2 = (1+i)$  and for all  $j \in \{1, 2, 3, 4\}$ ,  $\mathcal{H}_j^2 = (\pi_j)$ , so for all  $j \in \{0, 1, 2, 3, 4\}$ ,  $[\mathcal{H}_j]$  is a strongly ambiguous class of  $\mathbb{k}/F$  i.e.  $[\mathcal{H}_j] \in Am_s(\mathbb{k}/F)$ .

**Proposition 11.** *Let  $d = 2p_1p_2$  and  $\mathbb{k} = \mathbb{Q}(\sqrt{d}, i)$ .*

- (1)  $\mathbb{k}^{(*)} \subsetneq (\mathbb{k}/F)^*$ .
- (2) Assume that  $(p_1 \equiv p_2 \equiv 1 \pmod{8} \text{ and } Q_{\mathbb{k}} = 1) \text{ or } (p_1 \equiv 5 \text{ or } p_2 \equiv 5 \pmod{8})$ , then
- (i) If  $N(\varepsilon_d) = -1$ , then  $Am_s(\mathbb{k}/\mathbb{Q}(i)) = \langle [\mathcal{H}_0], [\mathcal{H}_1], [\mathcal{H}_2] \rangle = \langle [\mathcal{H}_0], [\mathcal{H}_3], [\mathcal{H}_4] \rangle$ .
  - (ii) Else  $Am_s(\mathbb{k}/\mathbb{Q}(i)) = \langle [\mathcal{H}_0], [\mathcal{H}_1], [\mathcal{H}_3] \rangle$ .
- (3) If  $p_1 \equiv p_2 \equiv 1 \pmod{8} \text{ and } Q_{\mathbb{k}} = 2$ , then
- $$Am_s(\mathbb{k}/\mathbb{Q}(i)) = \langle [\mathcal{H}_0], [\mathcal{H}_1], [\mathcal{H}_2], [\mathcal{H}_3] \rangle.$$

*Proof.* (1) As  $\mathbb{k} = \mathbb{Q}(\sqrt{2p_1p_2}, i)$ , so  $[\mathbb{k}^{(*)} : \mathbb{k}] = 4$ ; moreover, according to [14, Proposition 2, p.90],  $r = \text{rank} \mathbf{C}_{\mathbb{k},2} = 4$  if  $p_1 \equiv p_2 \equiv 1 \pmod{8}$  and  $r = \text{rank} \mathbf{C}_{\mathbb{k},2} = 3$  if  $p_1 \equiv 5 \text{ or } p_2 \equiv 5 \pmod{8}$ , so  $[(\mathbb{k}/F)^* : \mathbb{k}] = 8 \text{ or } 16$ , which implies the result.

(2) Note first that if  $p_1 \equiv 5 \text{ or } p_2 \equiv 5 \pmod{8}$ , then Lemma 6 states that  $Q_{\mathbb{k}} = 1$ , hence  $N(\varepsilon_d) = -1$  or  $(N(\varepsilon_d) = 1 \text{ and } x \pm 1 \text{ is not a square in } \mathbb{N})$  (see Lemma 5), where  $\varepsilon_d = x + y\sqrt{2p_1p_2}$ .

(i) Since  $(\mathcal{H}_1\mathcal{H}_2)^2 = (p_1)$ ,  $(\mathcal{H}_3\mathcal{H}_4)^2 = (p_2)$ , so Proposition 10 implies that  $\mathcal{H}_1\mathcal{H}_2$ ,  $\mathcal{H}_3\mathcal{H}_4$  are not principal in  $\mathbb{k}$ , hence  $\mathcal{H}_1$  and  $\mathcal{H}_2$  (resp.  $\mathcal{H}_3$  and  $\mathcal{H}_4$ ) lie in different classes. Moreover Lemma 9 states that  $[\mathcal{H}_0\mathcal{H}_1] = [\mathcal{H}_3]$  and  $[\mathcal{H}_0\mathcal{H}_2] = [\mathcal{H}_4]$  or  $[\mathcal{H}_0\mathcal{H}_2] = [\mathcal{H}_3]$  and  $[\mathcal{H}_0\mathcal{H}_1] = [\mathcal{H}_4]$ . Since  $(\mathcal{H}_0\mathcal{H}_1\mathcal{H}_2)^2 = ((1+i)p_1)$  (resp.  $(\mathcal{H}_0\mathcal{H}_3\mathcal{H}_4)^2 = ((1+i)p_2)$ ) and  $\sqrt{2} \notin \mathbb{Q}(\sqrt{2p_1p_2})$ , then Proposition 9 yields that  $\mathcal{H}_0\mathcal{H}_1\mathcal{H}_2$  and  $\mathcal{H}_0\mathcal{H}_3\mathcal{H}_4$  are not principal in  $\mathbb{k}$ . Therefore

$$\langle [\mathcal{H}_0], [\mathcal{H}_1], [\mathcal{H}_2] \rangle \subseteq Am_s(\mathbb{k}/\mathbb{Q}(i)) \text{ and } \langle [\mathcal{H}_0], [\mathcal{H}_3], [\mathcal{H}_4] \rangle \subseteq Am_s(\mathbb{k}/\mathbb{Q}(i)).$$

On the other hand, since  $Q_{\mathbb{k}} = 1$ , then by Lemma 5 we get  $E_{\mathbb{k}} = \langle i, \varepsilon_3 \rangle$ , so  $N_{\mathbb{k}/F}(E_{\mathbb{k}}) = \langle -1 \rangle$ .

(a) Suppose  $p_1 \equiv p_2 \equiv 1 \pmod{8}$ , so  $r = \text{rank} \mathbf{C}_{\mathbb{k},2} = 4$  and it is well known that  $i$  is norm in  $\mathbb{k}/F$ , hence  $|Am(\mathbb{k}/F)| = 2^4$  and  $E_F \cap N_{\mathbb{k}/F}(\mathbb{k}^\times) = E_F = \langle i \rangle$ , therefore formula (13) yields that  $|Am_s(\mathbb{k}/F)| = 8$ , this states that

$$Am_s(\mathbb{k}/\mathbb{Q}(i)) = \langle [\mathcal{H}_0], [\mathcal{H}_1], [\mathcal{H}_2] \rangle = \langle [\mathcal{H}_0], [\mathcal{H}_3], [\mathcal{H}_4] \rangle.$$

(b) Suppose  $p_1 \equiv 5 \text{ or } p_2 \equiv 5 \pmod{8}$ , so  $r = \text{rank} \mathbf{C}_{\mathbb{k},2} = 3$  and in this case  $i$  is not norm in  $\mathbb{k}/F$ , therefore  $|Am(\mathbb{k}/F)| = 2^3$  and  $E_F \cap N_{\mathbb{k}/F}(\mathbb{k}^\times) = \langle -1 \rangle$ , so formula (13) yields that  $|Am(\mathbb{k}/F)| = |Am_s(\mathbb{k}/F)| = 8$ , and the result derived.

(ii) If  $N(\varepsilon_3) = 1$  and  $x \pm 1$  is not a square in  $\mathbb{N}$ , then from Lemma 4 we get  $p_1(x \pm 1)$  and  $2p_2(x \mp 1)$  or  $p_2(x \pm 1)$  and  $2p_1(x \mp 1)$  are squares in  $\mathbb{N}$ , hence Proposition 10 implies that  $\mathcal{H}_1\mathcal{H}_2$  and  $\mathcal{H}_3\mathcal{H}_4$  are principal in  $\mathbb{k}$ , since  $(\mathcal{H}_1\mathcal{H}_2)^2 = (p_1)$  and  $(\mathcal{H}_3\mathcal{H}_4)^2 = (p_2)$ , so  $[\mathcal{H}_1] = [\mathcal{H}_2]$  and  $[\mathcal{H}_3] = [\mathcal{H}_4]$ . Since

$$\begin{aligned} (\mathcal{H}_0\mathcal{H}_1\mathcal{H}_3)^2 &= ((1+i)\pi_1\pi_3), \\ (\mathcal{H}_1\mathcal{H}_3)^2 &= (\pi_1\pi_3), \end{aligned}$$

$$\begin{aligned} (1+i)\pi_1\pi_3 &= [(eg - 4fh) - 2(eh + fg)] + i[(eg - 4fh) + 2(eh + fg)], \\ \pi_1\pi_3 &= (eg - 4fh) + 2i(eh + fg), \\ [(eg - 4fh) - 2(eh + fg)]^2 + [(eg - 4fh) + 2(eh + fg)]^2 &= 2p_1p_2 \text{ and} \\ (eg - 4fh)^2 + 4(eh + fg)^2 &= p_1p_2, \end{aligned}$$

then Proposition 9 implies that  $\mathcal{H}_0\mathcal{H}_1\mathcal{H}_3$  and  $\mathcal{H}_1\mathcal{H}_3$  are not principal in  $\mathbb{k}$ , therefore

$$\langle [\mathcal{H}_0], [\mathcal{H}_1], [\mathcal{H}_3] \rangle \subseteq Am_s(\mathbb{k}/\mathbb{Q}(i)).$$

Proceeding as above we prove that  $|Am_s(\mathbb{k}/F)| = 8$ , which yields that

$$Am_s(\mathbb{k}/\mathbb{Q}(i)) = \langle [\mathcal{H}_0], [\mathcal{H}_1], [\mathcal{H}_3] \rangle.$$

(3) Since  $Q_{\mathbb{k}} = 2$ , then, from Lemma 5,  $E_{\mathbb{k}} = \langle i, \sqrt{i\varepsilon_3} \rangle$ ,  $N(\varepsilon_3) = 1$  and  $x \pm 1$  is a square in  $\mathbb{N}$ ; so, as  $(\mathcal{H}_1\mathcal{H}_2)^2 = (p_1)$  and  $(\mathcal{H}_3\mathcal{H}_4)^2 = (p_2)$ , Proposition 10 yields that  $[\mathcal{H}_1] \neq [\mathcal{H}_2]$  and  $[\mathcal{H}_3] \neq [\mathcal{H}_4]$ ; moreover, as  $(\mathcal{H}_0^2\mathcal{H}_1\mathcal{H}_2\mathcal{H}_3\mathcal{H}_4)^2 = (2p_1p_2)$ , then  $\mathcal{H}_1\mathcal{H}_2\mathcal{H}_3\mathcal{H}_4 = (\frac{\sqrt{2p_1p_2}}{1+i})$ , hence  $[\mathcal{H}_4] = [\mathcal{H}_1\mathcal{H}_2\mathcal{H}_3]$ ; therefore Proposition 9 allowed us to state that

$$\langle [\mathcal{H}_0], [\mathcal{H}_1], [\mathcal{H}_2], [\mathcal{H}_3] \rangle \subseteq Am_s(\mathbb{k}/\mathbb{Q}(i)).$$

As above we prove that  $|Am(\mathbb{k}/F)| = 2^4$  and  $E_F \cap N_{\mathbb{k}/F}(\mathbb{k}^\times) = \langle i \rangle$ , hence formula (13) yields that  $|Am(\mathbb{k}/F)| = |Am_s(\mathbb{k}/F)| = 16$ , so

$$Am_s(\mathbb{k}/\mathbb{Q}(i)) = \langle [\mathcal{H}_0], [\mathcal{H}_1], [\mathcal{H}_2], [\mathcal{H}_3] \rangle.$$

□

## 5. Proof of the Main Theorem

We know that  $\mathbb{k} = \mathbb{Q}(\sqrt{2p_1p_2}, i)$  and its genus field is  $\mathbb{k}^{(*)} = \mathbb{Q}(\sqrt{2}, \sqrt{p_1}, \sqrt{p_2}, i)$ , so  $[\mathbb{k}^{(*)} : \mathbb{k}] = 4$  and there are three unramified quadratic extensions of  $\mathbb{k}$  abelian over  $\mathbb{Q}$  which are  $\mathbb{K}_1 = \mathbb{k}(\sqrt{p_1}) = \mathbb{Q}(\sqrt{p_1}, \sqrt{2p_2}, i)$ ,



$\mathbb{K}_2 = \mathbb{k}(\sqrt{p_2}) = \mathbb{Q}(\sqrt{p_2}, \sqrt{2p_1}, i)$  and  $\mathbb{K}_3 = \mathbb{k}(\sqrt{2}) = \mathbb{Q}(\sqrt{2}, \sqrt{p_1p_2}, i)$ ; let  $N_i$  denote the norm  $N_{\mathbb{K}_i/\mathbb{k}}$ . Let  $\mathbf{C}_{\mathbb{k}}$  (resp  $\mathbf{C}_{\mathbb{K}_i}$ ) be the ideal class group of  $\mathbb{k}$  (resp  $\mathbb{K}_i$ ), we denote by  $J_{\mathbb{K}_i}$  the homomorphism from  $\mathbf{C}_{\mathbb{k}}$  to  $\mathbf{C}_{\mathbb{K}_i}$  that maps to the class of an ideal  $\mathcal{I}$  of  $\mathbb{k}$  the class of the ideal generated by  $\mathcal{I}$  in  $\mathbb{K}_i$ . We keep the notations defined in the beginning of the preceding section. To prove the Main Theorem, we must study the capitulation problem of the 2-classes of  $\mathbb{k}$  in each  $\mathbb{K}_i$  and in  $\mathbb{k}^{(*)}$ .

**5.1. Capitulation in  $\mathbb{K}_1$ .** Let  $\varepsilon_1$  (resp  $\varepsilon_2, \varepsilon_3$ ) denote the fundamental unit of  $\mathbb{Q}(\sqrt{p_1})$  (resp.  $\mathbb{Q}(\sqrt{2p_2}), \mathbb{Q}(\sqrt{2p_1p_2})$ ). Put  $\varepsilon_3 = x + y\sqrt{2p_1p_2}$ .

**Theorem 3.** *Assume that  $N(\varepsilon_2) = N(\varepsilon_3) = 1$ , then*

- (1) *If  $x \pm 1$  is a square in  $\mathbb{N}$ , then  $|\ker J_{\mathbb{K}_1}| = 4$ .*
- (2) *Else  $|\ker J_{\mathbb{K}_1}| = 2$ .*

*Proof.* From Proposition 4,  $E_{\mathbb{K}_1} = \langle i, \varepsilon_1, \sqrt{i\varepsilon_2}, \sqrt{i\varepsilon_2\varepsilon_3} \rangle$  or  $E_{\mathbb{K}_1} = \langle i, \varepsilon_1, \sqrt{i\varepsilon_2}, \sqrt{i\varepsilon_3} \rangle$ , then  $N_1(E_{\mathbb{K}_1}) = \langle i, \varepsilon_3 \rangle$ .

(1) From Lemma 5,  $E_{\mathbb{k}} = \langle i, \sqrt{i\varepsilon_3} \rangle$ , so  $[E_{\mathbb{k}} : N_1(E_{\mathbb{K}_1})] = 2$ , and the relation (1) implies that  $|\ker J_{\mathbb{K}_1}| = 4$ .

(2) If  $x \pm 1$  is not a square in  $\mathbb{N}$ , then  $E_{\mathbb{k}} = \langle i, \varepsilon_3 \rangle$ , so  $[E_{\mathbb{k}} : N_1(E_{\mathbb{K}_1})] = 1$  and  $|\ker J_{\mathbb{K}_1}| = 2$ .  $\square$

**Corollary 1.** *We keep the assumptions of the preceding theorem.*

- (1) *If  $x \pm 1$  is a square in  $\mathbb{N}$ , then  $\ker J_{\mathbb{K}_1} = \langle [\mathcal{H}_1], [\mathcal{H}_2] \rangle \subset Am_s(\mathbb{k}/\mathbb{Q}(i))$ .*
- (2) *Else  $\ker J_{\mathbb{K}_1} = \langle [\mathcal{H}_1] \rangle \subset Am_s(\mathbb{k}/\mathbb{Q}(i))$ .*

*Proof.* As  $\mathcal{H}_1^2 = (\pi_1)$ ,  $\mathcal{H}_2^2 = (\pi_2)$  and  $\sqrt{e^2 + (2f)^2} = \sqrt{p_1} \notin \mathbb{Q}(\sqrt{2p_1p_2})$ , hence Proposition 9 implies that  $\mathcal{H}_1, \mathcal{H}_2$  are not principal in  $\mathbb{k}$ .

Let us prove that  $\mathcal{H}_1, \mathcal{H}_2$  capitulate in  $\mathbb{K}_1$ . According to the equalities (2),

$$\sqrt{2\pi_1\varepsilon_1} \in \mathbb{K}_1 \text{ and } \sqrt{2\pi_2\varepsilon_1} \in \mathbb{K}_1 \text{ or } \sqrt{\pi_1\varepsilon_1} \in \mathbb{K}_1 \text{ and } \sqrt{\pi_1\varepsilon_2} \in \mathbb{K}_1,$$

so putting

$$\sqrt{2\pi_1\varepsilon_1} = \alpha_1, \sqrt{2\pi_2\varepsilon_1} = \alpha_2, \sqrt{\pi_1\varepsilon_1} = \beta_1 \text{ and } \sqrt{\pi_1\varepsilon_2} = \beta_2,$$

we get

$$\mathcal{H}_1^2 = \left(\frac{\alpha_1}{1+i}\right)^2 \text{ and } \mathcal{H}_2^2 = \left(\frac{\alpha_2}{1+i}\right)^2 \text{ or } \mathcal{H}_1^2 = (\beta_1)^2 \text{ and } \mathcal{H}_2^2 = (\beta_2)^2$$

which implies the result.

(1) If  $x \pm 1$  is a square in  $\mathbb{N}$ , then  $p_1(x \pm 1)$  and  $2p_1(x \pm 1)$  are not squares in  $\mathbb{N}$ ; moreover as  $(p_1) = (\mathcal{H}_1\mathcal{H}_2)^2$ , then Proposition 10 yields that  $\mathcal{H}_1\mathcal{H}_2$  is not principal in  $\mathbb{k}$ , so  $\mathcal{H}_1$  and  $\mathcal{H}_2$  lie in different classes. Thus  $\ker J_{\mathbb{K}_1} = \langle [\mathcal{H}_1], [\mathcal{H}_2] \rangle$  which is a subgroup of  $Am_s(\mathbb{k}/\mathbb{Q}(i))$ .

(2) If  $x \pm 1$  is not a square in  $\mathbb{N}$ , then  $\mathcal{H}_1\mathcal{H}_2$  is principal in  $\mathbb{k}$ , so  $\mathcal{H}_1, \mathcal{H}_2$  lie in the same class; thus  $\ker J_{\mathbb{K}_1} = \langle [\mathcal{H}_1] \rangle \subset Am_s(\mathbb{k}/\mathbb{Q}(i))$ .  $\square$

**Numerical Examples** 1. (1)  $x \pm 1$  is a square in  $\mathbb{N}$ .

The first table gives integers  $d$  for which  $x \pm 1$  is a square in  $\mathbb{N}$ , when the second shows that for these integers  $\mathcal{H}_1\mathcal{H}_2$  is not principal in  $\mathbb{k}$  and  $\mathcal{H}_1, \mathcal{H}_2$  capitulate in  $\mathbb{K}_1$ .

$d = 2.p_1.p_2$	$\varepsilon_d = x + y\sqrt{d}$	$x + 1$	$x - 1$
$1394 = 2.41.17$	$12545 + 336\sqrt{1394}$	12546	12544
$3298 = 2.97.17$	$161603 + 2814\sqrt{3298}$	161604	161602
$15266 = 2.449.17$	$1236545 + 10008\sqrt{15266}$	1236546	1236544

$d = 2.p_1.p_2$	$\mathcal{H}_1\mathcal{H}_2$ in $\mathbb{k}$	$\mathcal{H}_1$ in $\mathbb{K}_1$	$\mathcal{H}_2$ in $\mathbb{K}_1$
$1394 = 2.41.17$	$[0, 0, 1, 0]$	$[0, 0, 0, 0]$	$[0, 0, 0, 0]$
$3298 = 2.97.17$	$[4, 2, 1, 0]$	$[0, 0, 0, 0]$	$[0, 0, 0, 0]$
$15266 = 2.449.17$	$[0, 0, 1, 0]$	$[0, 0, 0, 0]$	$[0, 0, 0, 0]$

(2)  $x + 1$  and  $x - 1$  are not squares in  $\mathbb{N}$ .

In this table we give integers  $d$  for which  $x + 1$  and  $x - 1$  are not squares in  $\mathbb{N}$ , we note that  $\mathcal{H}_1\mathcal{H}_2$  is principal in  $\mathbb{k}$  and  $\mathcal{H}_1$  capitulates in  $\mathbb{K}_1$ .

$d = 2.p_1.p_2$	$\varepsilon_d = x + y\sqrt{d}$	$x + 1$	$x - 1$	$\mathcal{H}_1\mathcal{H}_2$ in $\mathbb{k}$	$\mathcal{H}_1$ in $\mathbb{K}_1$
$890 = 2.5.89$	$179 + 6\sqrt{890}$	180	178	$[0, 0, 0]$	$[0, 0, 0, 0]$
$1802 = 2.53.17$	$849 + 20\sqrt{1802}$	850	848	$[0, 0, 0]$	$[0, 0, 0, 0]$
$5402 = 2.37.73$	$147 + 2\sqrt{5402}$	148	146	$[0, 0, 0]$	$[0, 0, 0]$

**Theorem 4.** (1) If  $N(\varepsilon_2) = N(\varepsilon_3) = -1$  or  $N(\varepsilon_2) = -N(\varepsilon_3) = 1$ , then  $|\ker J_{\mathbb{K}_1}| = 4$ .

- (2) Assume that  $N(\varepsilon_3) = -N(\varepsilon_2) = 1$ , then
- (i) If  $x \pm 1$  is a square in  $\mathbb{N}$ , then  $|\ker J_{\mathbb{K}_1}| = 8$ .
  - (ii) Else  $|\ker J_{\mathbb{K}_1}| = 4$ .

*Proof.* (1) From Propositions 1 and 2,  $N_1(E_{\mathbb{K}_1}) = \langle -1, \varepsilon_3 \rangle$  or  $\langle -1, i\varepsilon_3 \rangle$  or  $\langle i, \varepsilon_3^2 \rangle$ ; on the other hand, Lemma 5 yields that  $E_{\mathbb{K}} = \langle i, \varepsilon_3 \rangle$ , consequently  $[E_{\mathbb{K}} : N_1(E_{\mathbb{K}_1})] = 2$  and  $|\ker J_{\mathbb{K}_1}| = 4$ .

(2) From Proposition 3, we get:

- (i) If  $x \pm 1$  is a square in  $\mathbb{N}$ , then  $N_1(E_{\mathbb{K}_1}) = \langle -1, i\varepsilon_3 \rangle$ , as Lemma 5 yields that  $E_{\mathbb{K}} = \langle i, \sqrt{i\varepsilon_3} \rangle$ , so  $[E_{\mathbb{K}} : N_1(E_{\mathbb{K}_1})] = 4$  and  $|\ker J_{\mathbb{K}_1}| = 8$ .
- (ii) If  $x \pm 1$  is not a square in  $\mathbb{N}$ , then  $N_1(E_{\mathbb{K}_1}) = \langle -1, \varepsilon_3 \rangle$  or  $\langle -1, i\varepsilon_3 \rangle$ ; since in this case  $E_{\mathbb{K}} = \langle i, \varepsilon_3 \rangle$ , hence  $[E_{\mathbb{K}} : N_1(E_{\mathbb{K}_1})] = 2$  and  $|\ker J_{\mathbb{K}_1}| = 4$ .  $\square$

**Corollary 2.** (1) If  $N(\varepsilon_2) = N(\varepsilon_3) = -1$  or  $N(\varepsilon_2) = -N(\varepsilon_3) = 1$ , then  $\ker J_{\mathbb{K}_1} = \langle [\mathcal{H}_1], [\mathcal{H}_2] \rangle \subset Am_s(\mathbb{k}/\mathbb{Q}(i))$ .

(2) Assume that  $N(\varepsilon_3) = -N(\varepsilon_2) = 1$ , then

- (i) If  $x \pm 1$  is a square in  $\mathbb{N}$ , then  $\ker J_{\mathbb{K}_1} = \langle [\mathcal{H}_1], [\mathcal{H}_2], [\mathcal{H}_0\mathcal{H}_3] \rangle \subset Am_s(\mathbb{k}/\mathbb{Q}(i))$ .
- (ii) Else  $\ker J_{\mathbb{K}_1} = \langle [\mathcal{H}_1], [\mathcal{H}_0\mathcal{H}_3] \rangle \subset Am_s(\mathbb{k}/\mathbb{Q}(i))$ .

*Proof.* (1) As  $N(\varepsilon_3) = -1$ , then Proposition 10 yields that  $\mathcal{H}_1\mathcal{H}_2$  is not principal in  $\mathbb{k}$ ; even Proposition 9 implies that  $\mathcal{H}_1, \mathcal{H}_2$  are not principal in  $\mathbb{k}$ . Proceeding as in Corollary 1, we prove that  $\ker J_{\mathbb{K}_1} = \langle [\mathcal{H}_1], [\mathcal{H}_2] \rangle$ . With our assumptions,  $Am_s(\mathbb{k}/\mathbb{Q}(i)) = \langle [\mathcal{H}_0], [\mathcal{H}_1], [\mathcal{H}_2] \rangle$  (see Proposition 11), so  $\ker J_{\mathbb{K}_1}$  is a subgroup of  $Am_s(\mathbb{k}/\mathbb{Q}(i))$ .

(2) As  $(\mathcal{H}_0\mathcal{H}_1\mathcal{H}_3)^2 = ((1+i)\pi_1\pi_3)$ , so proceeding as in Proposition 11, we get  $\mathcal{H}_0\mathcal{H}_1\mathcal{H}_3, \mathcal{H}_0\mathcal{H}_2\mathcal{H}_3$  are not principal in  $\mathbb{k}$ . On the other hand, as  $N(\varepsilon_2) = -1$ , so the equalities (3) yields that  $\sqrt{(1 \pm i)\pi_3\varepsilon_2} \in \mathbb{K}_1$ , hence there exists  $\alpha \in \mathbb{K}_1$  such that  $(1 \pm i)\pi_3\varepsilon_2 = \alpha^2$  i.e.  $(\mathcal{H}_0\mathcal{H}_3)^2 = (\alpha^2)$ , therefore  $\mathcal{H}_0\mathcal{H}_3$  capitulates in  $\mathbb{K}_1$ .

(i) If  $x \pm 1$  is a square in  $\mathbb{N}$ , then Propositions 9 and 10 state that  $\mathcal{H}_1, \mathcal{H}_2$  and  $\mathcal{H}_1\mathcal{H}_2$  are not principal in  $\mathbb{k}$ . Hence  $\ker J_{\mathbb{K}_1} = \langle [\mathcal{H}_1], [\mathcal{H}_2], [\mathcal{H}_0\mathcal{H}_3] \rangle$ . By Proposition 11 (2), we get  $\ker J_{\mathbb{K}_1} \subset Am_s(\mathbb{k}/\mathbb{Q}(i))$ .

(ii) If  $x \pm 1$  is not a square in  $\mathbb{N}$ , then  $\mathcal{H}_1, \mathcal{H}_2$  lie in the same class; so  $\ker J_{\mathbb{K}_1} = \langle [\mathcal{H}_1], [\mathcal{H}_0\mathcal{H}_3] \rangle$ , and by Proposition 11 (1).(ii), we get  $\ker J_{\mathbb{K}_1} \subset Am_s(\mathbb{k}/\mathbb{Q}(i))$ .  $\square$

**Numerical Examples 2.** (1)  $N(\varepsilon_2) = N(\varepsilon_3) = -1$  or  $N(\varepsilon_2) = -N(\varepsilon_3) = 1$ .

This table gives integers  $d$  for which  $\mathcal{H}_1\mathcal{H}_2$  is not principal in  $\mathbb{k}$  and  $\mathcal{H}_1, \mathcal{H}_2$  capitulate in  $\mathbb{K}_1$ .

$d = 2.p_1.p_2$	$N(\varepsilon_2)$	$N(\varepsilon_3)$	$\mathcal{H}_1\mathcal{H}_2$ in $\mathbb{k}$	$\mathcal{H}_1$ in $\mathbb{K}_1$	$\mathcal{H}_2$ in $\mathbb{K}_1$
$290 = 2.5.29$	$-1$	$-1$	$[5, 0, 0]$	$[0, 0, 0]$	$[0, 0, 0]$
$442 = 2.13.17$	$1$	$-1$	$[2, 0, 0]$	$[0, 0, 0]$	$[0, 0, 0]$
$754 = 2.29.13$	$1$	$-1$	$[0, 0, 1]$	$[0, 0, 0]$	$[0, 0, 0]$
$1066 = 2.41.13$	$-1$	$-1$	$[0, 1, 0]$	$[0, 0, 0]$	$[0, 0, 0]$

(2)  $N(\varepsilon_3) = -N(\varepsilon_2) = 1$ .

(i)  $x \pm 1$  is a square in  $\mathbb{N}$ .

The first table gives integers  $d$  for which  $x \pm 1$  is a square in  $\mathbb{N}$  and  $\mathcal{H}_1\mathcal{H}_2$  is not principal in  $\mathbb{k}$ , when the second shows that for these integers  $\mathcal{H}_1, \mathcal{H}_2$  and  $\mathcal{H}_0\mathcal{H}_3$  capitulate in  $\mathbb{K}_1$ , but  $\mathcal{H}_0, \mathcal{H}_3$  do not.

$d = 2.p_1.p_2$	$\varepsilon_d = x + y\sqrt{d}$	$x + 1$	$x - 1$	$\mathcal{H}_1\mathcal{H}_2$ in $\mathbb{k}$
$1394 = 2.17.41$	$12545 + 336\sqrt{1394}$	12546	12544	$[0, 0, 1, 0]$
$7298 = 2.89.41$	$357603 + 4186\sqrt{7298}$	357604	357602	$[0, 0, 0, 1]$
$16498 = 2.73.113$	$1336337 + 10404\sqrt{16498}$	1336338	1336336	$[48, 0, 1, 1]$

$d = 2.p_1.p_2$	$\mathcal{H}_1$ in $\mathbb{K}_1$	$\mathcal{H}_2$ in $\mathbb{K}_1$	$\mathcal{H}_0\mathcal{H}_3$ in $\mathbb{K}_1$	$\mathcal{H}_3$ in $\mathbb{K}_1$	$\mathcal{H}_0$ in $\mathbb{K}_1$
$1394 = 2.17.41$	$[0, 0, 0, 0]$	$[0, 0, 0, 0]$	$[0, 0, 0, 0]$	$[24, 0, 0, 0]$	$[24, 0, 0, 0]$
$7298 = 2.89.41$	$[0, 0, 0, 0]$	$[0, 0, 0, 0]$	$[0, 0, 0, 0]$	$[42, 2, 2, 0]$	$[42, 2, 2, 0]$
$16498 = 2.73.113$	$[0, 0, 0, 0]$	$[0, 0, 0, 0]$	$[0, 0, 0, 0]$	$[96, 0, 0, 0]$	$[96, 0, 0, 0]$

(ii)  $x + 1$  and  $x - 1$  are not squares in  $\mathbb{N}$ .

The first table gives integers  $d$  for which  $x + 1$  and  $x - 1$  are not squares in  $\mathbb{N}$  and  $\mathcal{H}_1\mathcal{H}_2$  is principal in  $\mathbb{k}$ , when the second shows that for these integers  $\mathcal{H}_1$  and  $\mathcal{H}_0\mathcal{H}_3$  capitulate in  $\mathbb{K}_1$ , but  $\mathcal{H}_0, \mathcal{H}_3$  do not.

$d = 2.p_1.p_2$	$\varepsilon_d = x + y\sqrt{d}$	$x + 1$	$x - 1$	$\mathcal{H}_1\mathcal{H}_2$ in $\mathbb{k}$
$410 = 2.5.41$	$81 + 4\sqrt{410}$	82	80	$[0, 0, 0]$
$2938 = 2.13.113$	$786707 + 14514\sqrt{2938}$	786708	786706	$[0, 0, 0]$
$3034 = 2.37.41$	$4055973299 + 73635510\sqrt{3034}$	4055973300	4055973298	$[0, 0, 0]$
$8090 = 2.5.809$	$1619 + 18\sqrt{8090}$	1620	1618	$[0, 0, 0]$

$d = 2.p_1.p_2$	$\mathcal{H}_1$ in $\mathbb{K}_1$	$\mathcal{H}_0\mathcal{H}_3$ in $\mathbb{K}_1$	$\mathcal{H}_3$ in $\mathbb{K}_1$	$\mathcal{H}_0$ in $\mathbb{K}_1$
$410 = 2.5.41$	$[0, 0, 0]$	$[0, 0, 0]$	$[8, 0, 0]$	$[8, 0, 0]$
$2938 = 2.13.113$	$[0, 0, 0]$	$[0, 0, 0]$	$[24, 0, 0]$	$[24, 0, 0]$
$3034 = 2.37.41$	$[0, 0, 0]$	$[0, 0, 0]$	$[0, 0, 2]$	$[0, 0, 2]$
$8090 = 2.5.809$	$[0, 0, 0, 0]$	$[0, 0, 0, 0]$	$[168, 0, 0, 0]$	$[168, 0, 0, 0]$

**5.2. Capitulation in  $\mathbb{K}_2$ .** As  $p_1, p_2$  play symmetric roles, then putting  $\varepsilon_1$  (resp  $\varepsilon_2, \varepsilon_3$ ) the fundamental unit of  $\mathbb{Q}(\sqrt{p_2})$  (resp  $\mathbb{Q}(\sqrt{2p_1}), \mathbb{Q}(\sqrt{2p_1p_2})$ ) and proceeding as above, we get the following results. Put  $\varepsilon_3 = x + y\sqrt{2p_1p_2}$ .

**Theorem 5.** Assume that  $N(\varepsilon_2) = N(\varepsilon_3) = 1$ .

- (1) If  $x \pm 1$  is a square in  $\mathbb{N}$ , then  $|\ker J_{\mathbb{K}_2}| = 4$ .
- (2) Else  $|\ker J_{\mathbb{K}_2}| = 2$ .

**Corollary 3.** We keep the assumptions of the Theorem 5, then

- (1) If  $x \pm 1$  is a square in  $\mathbb{N}$ , then  $\ker J_{\mathbb{K}_2} = \langle [\mathcal{H}_3], [\mathcal{H}_4] \rangle \subset \text{Am}_s(\mathbb{k}/\mathbb{Q}(i))$ .
- (2) Else  $\ker J_{\mathbb{K}_2} = \langle [\mathcal{H}_3] \rangle \subset \text{Am}_s(\mathbb{k}/\mathbb{Q}(i))$ .

**Theorem 6.** (1) If  $N(\varepsilon_2) = N(\varepsilon_3) = -1$  or  $N(\varepsilon_2) = -N(\varepsilon_3) = 1$ , then

- $|\ker J_{\mathbb{K}_2}| = 4$ .
- (2) If  $N(\varepsilon_3) = -N(\varepsilon_2) = 1$ , then
  - (i) If  $x \pm 1$  is a square in  $\mathbb{N}$ , then  $|\ker J_{\mathbb{K}_2}| = 8$ .
  - (ii) Else  $|\ker J_{\mathbb{K}_2}| = 4$ .

**Corollary 4.** (1) If  $N(\varepsilon_2) = N(\varepsilon_3) = -1$  or  $N(\varepsilon_2) = -N(\varepsilon_3) = 1$ ,

then  $\ker J_{\mathbb{K}_2} = \langle [\mathcal{H}_3], [\mathcal{H}_4] \rangle \subset \text{Am}_s(\mathbb{k}/\mathbb{Q}(i))$ .

(2) Assume that  $N(\varepsilon_3) = -N(\varepsilon_2) = 1$ , then

- (i) If  $x \pm 1$  is a square in  $\mathbb{N}$ , then
 $\ker J_{\mathbb{K}_2} = \langle [\mathcal{H}_0\mathcal{H}_1], [\mathcal{H}_3], [\mathcal{H}_4] \rangle \subset \text{Am}_s(\mathbb{k}/\mathbb{Q}(i))$ .
- (ii) Else  $\ker J_{\mathbb{K}_2} = \langle [\mathcal{H}_0\mathcal{H}_1], [\mathcal{H}_3] \rangle \subset \text{Am}_s(\mathbb{k}/\mathbb{Q}(i))$ .

**5.3. Capitulation in  $\mathbb{K}_3$ .** Let  $\varepsilon_1$  (resp.  $\varepsilon_2, \varepsilon_3$ ) denote the fundamental unit of  $\mathbb{Q}(\sqrt{2})$  (resp.  $\mathbb{Q}(\sqrt{p_1p_2}), \mathbb{Q}(\sqrt{2p_1p_2})$ ) and  $q = q(\mathbb{K}_3^+/\mathbb{Q})$ .

**Theorem 7.** Suppose  $N(\varepsilon_2) = N(\varepsilon_3) = -1$  or  $N(\varepsilon_2) = -N(\varepsilon_3) = 1$ , then

- (1) If  $q = 1$ , then  $|\ker J_{\mathbb{K}_3}| = 4$ .
- (2) If  $q = 2$ , then  $|\ker J_{\mathbb{K}_3}| = 2$ .

*Proof.* Note first that  $\sqrt{i} = \frac{1+i}{\sqrt{2}} \in \mathbb{K}_3$ .

(1) From Propositions 5 and 6, we get  $E_{\mathbb{K}_3} = \langle \sqrt{i}, \varepsilon_1, \varepsilon_2, \varepsilon_3 \rangle$ , so  $N_3(E_{\mathbb{K}_3}) = \langle i, \varepsilon_3^2 \rangle$ ; as, from Lemma 5,  $E_{\mathbb{k}} = \langle i, \varepsilon_3 \rangle$ , then  $[E_{\mathbb{k}} : N_3(E_{\mathbb{K}_3})] = 2$  and  $|\ker J_{\mathbb{K}_3}| = 4$ .

(2) Proposition 6 yields that  $E_{\mathbb{K}_3} = \langle \sqrt{i}, \varepsilon_1, \varepsilon_2, \sqrt{\varepsilon_1 \varepsilon_2 \varepsilon_3} \rangle$ , so  $N_3(E_{\mathbb{K}_3}) = \langle i, \varepsilon_3 \rangle$ ; on the other hand, Lemma 5 implies that  $E_{\mathbb{k}} = \langle i, \varepsilon_3 \rangle$ , hence  $[E_{\mathbb{k}} : N_3(E_{\mathbb{K}_3})] = 1$  and  $|\ker J_{\mathbb{K}_3}| = 2$ .  $\square$

**Corollary 5.** *Keep the assumptions of Theorem 7, then*

(1) *If  $q = 2$ , then  $\ker J_{\mathbb{K}_3} = \langle [\mathcal{H}_0] \rangle \subset Am_s(\mathbb{k}/\mathbb{Q}(i))$ .*

(2) *If  $q = 1$ , then  $\ker J_{\mathbb{K}_3} = \langle [\mathcal{H}_0], [\mathcal{H}_1 \mathcal{H}_2] \rangle \subset Am_s(\mathbb{k}/\mathbb{Q}(i))$ .*

*Proof.* Note that  $\sqrt{(1+i)\varepsilon_1} = \frac{1}{2}(2 + (1+i)\sqrt{2})$ , so there exists  $\beta \in \mathbb{K}_3$  such that  $\mathcal{H}_0^2 = (1+i) = (\beta^2)$ , hence  $\mathcal{H}_0$  capitulates in  $\mathbb{K}_3$ .

(1) If  $q = 2$ , then  $\ker J_{\mathbb{K}_3} = \langle [\mathcal{H}_0] \rangle$ , hence by Proposition 11 (1).(i),  $\ker J_{\mathbb{K}_3} \subset Am_s(\mathbb{k}/\mathbb{Q}(i))$ .

(2) There are two cases to distinguish.

(i) If  $N(\varepsilon_2) = 1$ , then, from the proof of Proposition 5, we get that  $\sqrt{p_1 \varepsilon_2} \in \mathbb{K}_3$ , so putting  $\alpha = \sqrt{p_1 \varepsilon_3}$ , we deduce that  $(\alpha^2) = \mathcal{H}_1^2 \mathcal{H}_2^2$ , which implies that  $\mathcal{H}_1 \mathcal{H}_2$  capitulates in  $\mathbb{K}_3$ .

(ii) If  $N(\varepsilon_2) = -1$ , then  $\varepsilon_1 \varepsilon_2 \varepsilon_3$  is not a square in  $\mathbb{Q}(\sqrt{2}, \sqrt{p_1 p_2})$ , so, according to the remark 3, we get  $\sqrt{p_1 \varepsilon_1 \varepsilon_2 \varepsilon_3} \in \mathbb{K}_3$ , hence  $\mathcal{H}_1 \mathcal{H}_2$  capitulates in  $\mathbb{K}_3$ .

On the other hand, Proposition 10 yields that  $\mathcal{H}_1 \mathcal{H}_2$  is not principal in  $\mathbb{k}$ , furthermore  $(\mathcal{H}_0 \mathcal{H}_1 \mathcal{H}_2)^2 = ((1+i)p_1)$  and  $\sqrt{2} \notin \mathbb{k}$ , then Proposition 9 implies that  $\mathcal{H}_0 \mathcal{H}_1 \mathcal{H}_2$  is not principal in  $\mathbb{k}$  i.e.  $\mathcal{H}_0$  and  $\mathcal{H}_1 \mathcal{H}_2$  lie in different classes; hence the result. By Proposition 11 (1).(i) we get  $\ker J_{\mathbb{K}_3} \subset Am_s(\mathbb{k}/\mathbb{Q}(i))$ .  $\square$

**Numerical Examples 3.** (1) First case  $q = 2$ .

$d = 2.p_1.p_2$	$q$	$N(\varepsilon_2)$	$N(\varepsilon_3)$	$\mathcal{H}_0$ in $\mathbb{K}_3$
$130 = 2.5.13$	2	-1	-1	$[0, 0]$
$1066 = 2.13.41$	2	-1	-1	$[0, 0, 0, 0]$
$2146 = 2.29.37$	2	-1	-1	$[0, 0]$

(2) Second case  $q = 1$ .

$d = 2.p_1.p_2$	$q$	$N(\varepsilon_2)$	$N(\varepsilon_3)$	$\mathcal{H}_0$ in $\mathbb{K}_3$	$\mathcal{H}_1\mathcal{H}_2$ in $\mathbb{K}_3$	$\mathcal{H}_1\mathcal{H}_2$ in $\mathbb{k}$
$290 = 2.5.29$	1	-1	-1	[0, 0]	[0, 0]	[5, 0, 0]
$754 = 2.13.29$	1	1	-1	[0, 0]	[0, 0]	[0, 0, 1]
$962 = 2.37.13$	1	-1	-1	[0, 0]	[0, 0]	[7, 1, 1]
$1378 = 2.53.13$	1	1	-1	[0, 0]	[0, 0]	[5, 0, 0]

**Theorem 8.** *If  $N(\varepsilon_3) = -N(\varepsilon_2) = 1$ , then  $|\ker J_{\mathbb{K}_3}| = 4$ .*

*Proof.* If  $\varepsilon_3 = x + y\sqrt{2p_1p_2}$ , then from Proposition 7 we get:

- If  $x \pm 1$  is a square in  $\mathbb{N}$ , then  $N_3(E_{\mathbb{K}_3}) = \langle i, \varepsilon_3 \rangle$ ; as  $E_{\mathbb{k}} = \langle i, \sqrt{i\varepsilon_3} \rangle$ , so  $[E_{\mathbb{k}} : N_3(E_{\mathbb{K}_3})] = 2$  and  $|\ker J_{\mathbb{K}_3}| = 4$ .
- If  $x \pm 1$  is not a square in  $\mathbb{N}$ , then  $N_3(E_{\mathbb{K}_3}) = \langle i, \varepsilon_3^2 \rangle$ ; as  $E_{\mathbb{k}} = \langle i, \varepsilon_3 \rangle$ , hence  $[E_{\mathbb{k}} : N_3(E_{\mathbb{K}_3})] = 2$  and  $|\ker J_{\mathbb{K}_3}| = 4$ .  $\square$

**Corollary 6.** *Keep the assumptions of Theorem 8, then*

$$\ker J_{\mathbb{K}_3} = \langle [\mathcal{H}_0], [\mathcal{H}_1\mathcal{H}_3] \rangle \subset \text{Am}_s(\mathbb{k}/\mathbb{Q}(i)).$$

*Proof.* We have proved that  $\mathcal{H}_0$  capitulates in  $\mathbb{K}_3$ .

$\mathcal{H}_1\mathcal{H}_3$  capitulates in  $\mathbb{K}_3$  if and only if there exist a unit  $\varepsilon \in \mathbb{K}_3$ ,  $\alpha \in \mathbb{K}_3$  such that

$$\alpha^2 = \varepsilon\pi_1\pi_3,$$

as  $\pi_1\pi_3 = (e + 2if)(g + 2ih) = (eg - 4fh) + 2i(fg + eh)$ , so putting  $\alpha = \alpha_1 + i\alpha_2$ , where  $\alpha_j \in \mathbb{Q}(\sqrt{2}, \sqrt{p_1p_2})$ , and choosing  $\varepsilon$  real, we get

$$\alpha_1^2 - \alpha_2^2 = \varepsilon(eg - 4fh) \text{ and } \alpha_1\alpha_2 = \varepsilon(fg + eh),$$

hence

$$\alpha_1^4 - \varepsilon(eg - 4fh)\alpha_1^2 - \varepsilon^2(fg + eh)^2 = 0,$$

since  $p_1p_2 = (eg - 4fh)^2 + 4(fg + eh)^2$ , so

$$\alpha_1^2 = \frac{\varepsilon}{2}[(eg - 4fh) + \sqrt{p_1p_2}] = \frac{\varepsilon}{4}[\pi_1\pi_3 + \pi_2\pi_4 + 2\sqrt{p_1p_2}],$$

therefore

$$\alpha_1 = \frac{\sqrt{\varepsilon}}{2}[\sqrt{\pi_1\pi_3} + \sqrt{\pi_2\pi_4}].$$

Then if  $\varepsilon = \varepsilon_2$  and  $\sqrt{\varepsilon_2} = y_1\sqrt{\pi_1\pi_3} + y_2\sqrt{\pi_2\pi_4}$ , we get

$$\alpha_1 = \frac{1}{2}(a + b\sqrt{p_1p_2}) \text{ and } \alpha_2 = \frac{\varepsilon(fg + eh)}{\alpha_1},$$

where  $a$  and  $b$  are in  $\mathbb{Q}$ ; which implies that  $\mathcal{H}_1\mathcal{H}_3$  capitulates in  $\mathbb{K}_3$ .

On the other hand,  $\mathcal{H}_0$  and  $\mathcal{H}_1\mathcal{H}_3$  lie in different classes, in fact

$$\begin{aligned} (\mathcal{H}_0\mathcal{H}_1\mathcal{H}_3)^2 &= ((1+i)\pi_1\pi_3) = \\ &[(eg - 4fh) - 2(eh + fg)] + i[(eg - 4fh) + 2(eh + fg)] \\ \text{and } [(eg - 4fh) - 2(eh + fg)]^2 &+ [(eg - 4fh) + 2(eh + fg)]^2 = 2p_1p_2, \end{aligned}$$

so Proposition 9 yields the result. By Proposition 11 (1).(ii) and (2) we get  $\ker J_{\mathbb{K}_3} \subset \text{Am}_s(\mathbb{K}/\mathbb{Q}(i))$ .  $\square$

**Numerical Examples** 4. For the case:  $N(\varepsilon_3) = -N(\varepsilon_2) = 1$ ,  $\mathcal{H}_0$ ,  $\mathcal{H}_1\mathcal{H}_3$  capitulate in  $\mathbb{K}_3$  and  $\mathcal{H}_1$ ,  $\mathcal{H}_3$  do not.

$d = 2.p_1.p_2$	$\mathcal{H}_0$ in $\mathbb{K}_3$	$\mathcal{H}_1$ in $\mathbb{K}_3$	$\mathcal{H}_3$ in $\mathbb{K}_3$	$\mathcal{H}_1\mathcal{H}_3$ in $\mathbb{K}_3$
$890 = 2.5.89$	$[0, 0, 0]$	$[6, 0, 2]$	$[6, 0, 2]$	$[0, 0, 0]$
$1802 = 2.53.17$	$[0, 0, 0]$	$[48, 0, 0]$	$[48, 0, 0]$	$[0, 0, 0]$
$5002 = 2.61.41$	$[0, 0, 0, 0]$	$[112, 0, 0, 0]$	$[112, 0, 0, 0]$	$[0, 0, 0, 0]$

**Theorem 9.** Suppose  $N(\varepsilon_2) = N(\varepsilon_3) = 1$  and put  $\varepsilon_3 = x + y\sqrt{2p_1p_2}$ , then

- (1) If  $x \pm 1$  is a square in  $\mathbb{N}$ , then  $|\ker J_{\mathbb{K}_3}| = 4$ .
- (2) Else  $|\ker J_{\mathbb{K}_3}| = 2$ .

*Proof.* From Proposition 8, we get

- (1)  $N_3(E_{\mathbb{K}_3}) = \langle i, \varepsilon_3 \rangle$ ; as  $E_{\mathbb{K}} = \langle i, \sqrt{i\varepsilon_3} \rangle$ , so  $[E_{\mathbb{K}} : N_3(E_{\mathbb{K}_3})] = 2$  and  $|\ker J_{\mathbb{K}_3}| = 4$ .
- (2)  $N_3(E_{\mathbb{K}_3}) = \langle i, \varepsilon_3 \rangle$ ; since  $E_{\mathbb{K}} = \langle i, \varepsilon_3 \rangle$ , hence  $[E_{\mathbb{K}} : N_3(E_{\mathbb{K}_3})] = 1$  and  $|\ker J_{\mathbb{K}_3}| = 2$ .  $\square$

**Corollary 7.** We keep the assumptions of Theorem 9, then

- (1) If  $x \pm 1$  is a square in  $\mathbb{N}$ , then  $\ker J_{\mathbb{K}_3} = \langle [\mathcal{H}_0], [\mathcal{H}_1\mathcal{H}_2] \rangle \subset \text{Am}_s(\mathbb{K}/\mathbb{Q}(i))$ .
- (2) Else  $\ker J_{\mathbb{K}_3} = \langle [\mathcal{H}_0] \rangle \subset \text{Am}_s(\mathbb{K}/\mathbb{Q}(i))$ .

*Proof.* Note first that  $\mathcal{H}_0$  capitulates in  $\mathbb{K}_3$ .

(1) Proposition 10 states that  $\mathcal{H}_1\mathcal{H}_2$  is not principal in  $\mathbb{K}$ ; furthermore, from Equality (8), we get  $p_1\varepsilon_2$  is a square in  $\mathbb{K}_3$ , so  $\mathcal{H}_1\mathcal{H}_2$  capitulates in  $\mathbb{K}_3$ . As above we prove that  $\mathcal{H}_0$ ,  $\mathcal{H}_1\mathcal{H}_3$  lie in different classes, so  $\ker J_{\mathbb{K}_3} = \langle [\mathcal{H}_0], [\mathcal{H}_1\mathcal{H}_2] \rangle$ , and from proposition 11,  $\ker J_{\mathbb{K}_3} \subset \text{Am}_s(\mathbb{K}/\mathbb{Q}(i))$ .



(2) As  $\mathcal{H}_0$  capitulates in  $\mathbb{K}_3$ , so  $\ker J_{\mathbb{K}_3} = \langle [\mathcal{H}_0] \rangle$  which is a subgroup of  $\text{Am}_s(\mathbb{K}/\mathbb{Q}(i))$ .  $\square$

**Numerical Examples 5.** (1)  $x \pm 1$  is a square in  $\mathbb{N}$ . The first table gives integers  $d$  for which  $x \pm 1$  is square in  $\mathbb{N}$  and  $\mathcal{H}_1\mathcal{H}_2$  is not principal in  $\mathbb{k}$ , when the second shows that for these integers  $\mathcal{H}_0$  and  $\mathcal{H}_1\mathcal{H}_2$  capitulate in  $\mathbb{K}_3$ , but  $\mathcal{H}_1, \mathcal{H}_2$  do not.

$d = 2.p_1.p_2$	$\varepsilon_d = x + y\sqrt{d}$	$x + 1$	$x - 1$	$\mathcal{H}_1\mathcal{H}_2$ in $\mathbb{k}$
$12994 = 2.73.89$	$12995 + 114\sqrt{12994}$	12996	12994	$[0, 6, 2, 0]$
$14722 = 2.17.433$	$132497 + 1092\sqrt{14722}$	132498	132496	$[8, 0, 0, 0]$
$32882 = 2.41.401$	$295937 + 1632\sqrt{32882}$	295938	295936	$[48, 0, 0, 0]$
$46658 = 2.41.569$	$46657 + 216\sqrt{46658}$	46658	46656	$[64, 0, 0, 0]$

$d = 2.p_1.p_2$	$\mathcal{H}_0$ in $\mathbb{K}_3$	$\mathcal{H}_1$ in $\mathbb{K}_3$	$\mathcal{H}_2$ in $\mathbb{K}_3$	$\mathcal{H}_1\mathcal{H}_2$ in $\mathbb{K}_3$
$12994 = 2.73.89$	$[0, 0, 0, 0, 0, 0]$	$[0, 0, 0, 1, 1, 1]$	$[0, 0, 0, 1, 1, 1]$	$[0, 0, 0, 0, 0, 0]$
$14722 = 2.17.433$	$[0, 0, 0, 0, 0, 0]$	$[56, 4, 0, 0, 0, 0]$	$[56, 4, 0, 0, 0, 0]$	$[0, 0, 0, 0, 0, 0]$
$32882 = 2.41.401$	$[0, 0, 0, 0, 0, 0]$	$[0, 4, 0, 0, 0, 0]$	$[0, 4, 0, 0, 0, 0]$	$[0, 0, 0, 0, 0, 0]$
$46658 = 2.41.569$	$[0, 0, 0, 0, 0, 0]$	$[480, 8, 0, 0, 0, 0]$	$[480, 8, 0, 0, 0, 0]$	$[0, 0, 0, 0, 0, 0]$

(2)  $x + 1$  and  $x - 1$  are not squares in  $\mathbb{N}$ ,  $\mathcal{H}_0$  capitulates in  $\mathbb{K}_3$ .

$d = 2.p_1.p_2$	$\varepsilon_d = x + y\sqrt{d}$	$x + 1$	$x - 1$	$\mathcal{H}_0$ in $\mathbb{K}_3$
$410 = 2.5.41$	$81 + 4\sqrt{410}$	82	80	$[0, 0, 0, 0]$
$2938 = 2.13.113$	$786707 + 14514\sqrt{2938}$	786708	786706	$[0, 0, 0, 0]$
$4010 = 2.401.5$	$7219 + 114\sqrt{4010}$	7220	7218	$[0, 0, 0]$
$5402 = 2.37.73$	$147 + 2\sqrt{5402}$	148	146	$[0, 0, 0]$

#### 5.4. Capitulation in $\mathbb{k}^{(*)}$ .

**Theorem 10.** Let  $p_1 \equiv p_2 \equiv 1 \pmod{4}$  be primes and  $\mathbb{k}^{(*)}$  the genus field of  $\mathbb{k}$ . Let  $Q$  denote the unit index of the field  $\mathbb{Q}(\sqrt{2}, \sqrt{p_1 p_2})$  and  $\varepsilon_3$  the fundamental unit of  $\mathbb{Q}(\sqrt{2p_1 p_2})$ .

(1) If  $N(\varepsilon_3) = -1$ , then  $\text{Am}_s(\mathbb{k}/\mathbb{Q}(i)) = \langle \mathcal{H}_0, \mathcal{H}_1, \mathcal{H}_2 \rangle \subseteq \ker J_{\mathbb{k}^{(*)}}$ .

(2) If  $N(\varepsilon_3) = 1$ , then

(i) If  $Q = 2$ , then  $\text{Am}_s(\mathbb{k}/\mathbb{Q}(i)) = \langle \mathcal{H}_0, \mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3 \rangle \subseteq \ker J_{\mathbb{k}^{(*)}}$ .

(ii) If  $Q = 1$ , then  $\text{Am}_s(\mathbb{k}/\mathbb{Q}(i)) = \langle \mathcal{H}_0, \mathcal{H}_1, \mathcal{H}_3 \rangle \subseteq \ker J_{\mathbb{k}^{(*)}}$ .

*Proof.* (1) Propositions 9, 10 imply that the classes of  $\mathcal{H}_i$ , where  $i \in \{0, 1, 2, 3, 4\}$ , are pairwise different. On the other hand, from Corollaries 2, 4, 5 and 7 we

infer that  $[\mathcal{H}_i] \in \ker J_{\mathbb{k}^{(*)}}$ ; furthermore Lemma 9 states that  $[\mathcal{H}_3] = [\mathcal{H}_0\mathcal{H}_1]$  and  $[\mathcal{H}_4] = [\mathcal{H}_0\mathcal{H}_2]$  or  $[\mathcal{H}_4] = [\mathcal{H}_0\mathcal{H}_1]$  and  $[\mathcal{H}_3] = [\mathcal{H}_0\mathcal{H}_2]$ , hence Proposition 11 yields that  $Am_s(\mathbb{k}/\mathbb{Q}(i)) = \langle \mathcal{H}_0, \mathcal{H}_1, \mathcal{H}_2 \rangle \subseteq \ker J_{\mathbb{k}^{(*)}}$ .

(2) (i) If  $Q = 2$ , then Corollaries 1, 2, 3, 4 and 7 imply that  $\langle \mathcal{H}_0, \mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3, \mathcal{H}_4 \rangle \subseteq \ker J_{\mathbb{k}^{(*)}}$ . Moreover Propositions 9, 10 state that the ideals  $\mathcal{H}_1\mathcal{H}_2\mathcal{H}_3$ ,  $\mathcal{H}_i\mathcal{H}_j$  and  $\mathcal{H}_0\mathcal{H}_i\mathcal{H}_j$  are not principal in  $\mathbb{k}$  for all  $i \in \{1, 2\}$  and  $j \in \{3, 4\}$ ; on the other hand,  $(\mathcal{H}_0^2\mathcal{H}_1\mathcal{H}_2\mathcal{H}_3\mathcal{H}_4)^2 = (d)$ , so  $\mathcal{H}_1\mathcal{H}_2\mathcal{H}_3\mathcal{H}_4 = (\frac{\sqrt{d}}{1+i})$ , hence  $[\mathcal{H}_4] = [\mathcal{H}_1\mathcal{H}_2\mathcal{H}_3]$ , and the result derived.

(ii) If  $Q = 1$ , then  $\mathcal{H}_1$  and  $\mathcal{H}_2$  (resp.  $\mathcal{H}_3$  and  $\mathcal{H}_4$ ) lie in the same class, so from Corollaries 1, 2, 4 and 5, we get  $Am_s(\mathbb{k}/\mathbb{Q}(i)) = \langle \mathcal{H}_0, \mathcal{H}_1, \mathcal{H}_3 \rangle \subseteq \ker J_{\mathbb{k}^{(*)}}$ .

And this Completes the demonstration of the Main Theorem.  $\square$

**Numerical Examples 6.** (1) First case  $N(\varepsilon_d) = -1$ .

$d = 2.p_1.p_2$	$N(\varepsilon_d)$	$\mathcal{H}_0$ in $\mathbb{k}^{(*)}$	$\mathcal{H}_1$ in $\mathbb{k}^{(*)}$	$\mathcal{H}_2$ in $\mathbb{k}^{(*)}$
$442 = 2.13.17$	-1	$[0, 0, 0, 0, 0]$	$[0, 0, 0, 0, 0]$	$[0, 0, 0, 0, 0]$
$1066 = 2.41.13$	-1	$[0, 0, 0]$	$[0, 0, 0]$	$[0, 0, 0]$
$1258 = 2.17.37$	-1	$[0, 0]$	$[0, 0]$	$[0, 0]$

(2) Second case  $N(\varepsilon_d) = 1$ .

(i)  $Q = 2$ . The first table gives examples where  $\mathcal{H}_1\mathcal{H}_2$  and  $\mathcal{H}_3\mathcal{H}_4$  are not principal in  $\mathbb{k}$ , when the second shows that  $\mathcal{H}_0, \mathcal{H}_1, \mathcal{H}_2$  and  $\mathcal{H}_3$  capitulate in  $\mathbb{k}^{(*)}$ .

$d = 2.p_1.p_2$	$N(\varepsilon_d)$	$Q$	$\mathcal{H}_1\mathcal{H}_2$ in $\mathbb{k}$	$\mathcal{H}_3\mathcal{H}_4$ in $\mathbb{k}$
$1394 = 2.17.41$	1	2	$[0, 0, 1, 0]$	$[0, 0, 1, 0]$
$3298 = 2.97.17$	1	2	$[4, 2, 1, 0]$	$[4, 2, 1, 0]$
$3842 = 2.17.113$	1	2	$[0, 0, 1, 0]$	$[0, 0, 1, 0]$

$d = 2.p_1.p_2$	$\mathcal{H}_0$ in $\mathbb{k}^{(*)}$	$\mathcal{H}_1$ in $\mathbb{k}^{(*)}$	$\mathcal{H}_2$ in $\mathbb{k}^{(*)}$	$\mathcal{H}_3$ in $\mathbb{k}^{(*)}$
$1394 = 2.17.41$	$[0, 0, 0, 0, 0, 0]$	$[0, 0, 0, 0, 0, 0]$	$[0, 0, 0, 0, 0, 0]$	$[0, 0, 0, 0, 0, 0]$
$3298 = 2.97.17$	$[0, 0, 0, 0, 0, 0]$	$[0, 0, 0, 0, 0, 0]$	$[0, 0, 0, 0, 0, 0]$	$[0, 0, 0, 0, 0, 0]$
$3842 = 2.17.113$	$[0, 0, 0, 0, 0, 0]$	$[0, 0, 0, 0, 0, 0]$	$[0, 0, 0, 0, 0, 0]$	$[0, 0, 0, 0, 0, 0]$

(ii)  $Q = 1$ . The first table gives examples where  $\mathcal{H}_1\mathcal{H}_2$  and  $\mathcal{H}_3\mathcal{H}_4$  are principal in  $\mathbb{k}$ , when the second shows that  $\mathcal{H}_0, \mathcal{H}_1$  and  $\mathcal{H}_3$  capitulate in  $\mathbb{k}^{(*)}$ .

$d = 2.p_1.p_2$	$N(\varepsilon_d)$	$Q$	$\mathcal{H}_1\mathcal{H}_2$ in $\mathbb{k}$	$\mathcal{H}_3\mathcal{H}_4$ in $\mathbb{k}$
$890 = 2.5.89$	1	1	$[0, 0, 0]$	$[0, 0, 0]$
$1802 = 2.53.17$	1	1	$[0, 0, 0]$	$[0, 0, 0]$
$2938 = 2.13.113$	1	1	$[0, 0, 0]$	$[0, 0, 0]$

$d = 2.p_1.p_2$	$\mathcal{H}_0$ in $\mathbb{k}^{(*)}$	$\mathcal{H}_1$ in $\mathbb{k}^{(*)}$	$\mathcal{H}_3$ in $\mathbb{k}^{(*)}$
$890 = 2.5.89$	$[0, 0, 0, 0, 0]$	$[0, 0, 0, 0, 0]$	$[0, 0, 0, 0, 0]$
$1802 = 2.53.17$	$[0, 0, 0, 0, 0]$	$[0, 0, 0, 0, 0]$	$[0, 0, 0, 0, 0]$
$2938 = 2.13.113$	$[0, 0, 0, 0, 0]$	$[0, 0, 0, 0, 0]$	$[0, 0, 0, 0, 0]$

## 6. Application

Assume  $\mathbf{C}_{\mathbb{k},2}$  is of type  $(2, 2, 2)$ ; which occurs if and only if  $p_1 \equiv p_2 \equiv 1 \pmod{4}$  are primes and at least two elements of the set  $\{(\frac{p_1}{p_2}), (\frac{2}{p_1}), (\frac{2}{p_2})\}$  are equal to  $-1$  (see [5]). Under these assumptions the norm of the fundamental unit of  $\mathbb{Q}(\sqrt{2p_1p_2})$  is equal to  $-1$  and  $\mathbf{C}_{\mathbb{k},2} = Am_s(\mathbb{k}/\mathbb{Q}(i)) = \langle [\mathcal{H}_0], [\mathcal{H}_1], [\mathcal{H}_2] \rangle$  (see [6]).

**Theorem 11.** *Let  $\mathbb{k} = \mathbb{Q}(\sqrt{2p_1p_2}, i)$ . Put  $\mathbb{K}_1 = \mathbb{k}(\sqrt{p_1})$ ,  $\mathbb{K}_2 = \mathbb{k}(\sqrt{p_2})$ , then exactly four classes capitulate in  $\mathbb{K}_1$ ,  $\mathbb{K}_2$  and  $\ker J_{\mathbb{K}_1} = \langle [\mathcal{H}_1], [\mathcal{H}_2] \rangle$ ,  $\ker J_{\mathbb{K}_2} = \langle [\mathcal{H}_0\mathcal{H}_1], [\mathcal{H}_0\mathcal{H}_2] \rangle$ .*

*Proof.* Since  $N(\varepsilon_3) = -1$ , so from Theorems 4, 6 we get  $|\ker J_{\mathbb{K}_i}| = 4$ , where  $i \in \{1, 2\}$ . Corollaries 2, 4 imply that  $\ker J_{\mathbb{K}_1} = \langle [\mathcal{H}_1], [\mathcal{H}_2] \rangle$  and  $\ker J_{\mathbb{K}_2} = \langle [\mathcal{H}_3], [\mathcal{H}_4] \rangle$ , but as the norm of  $\varepsilon_3$  is equal to  $-1$ , then  $\mathcal{H}_0\mathcal{H}_1\mathcal{H}_3$  and  $\mathcal{H}_0\mathcal{H}_2\mathcal{H}_4$  or  $\mathcal{H}_0\mathcal{H}_2\mathcal{H}_3$  and  $\mathcal{H}_0\mathcal{H}_1\mathcal{H}_4$  are principal in  $\mathbb{k}$ , therefore  $\ker J_{\mathbb{K}_2} = \langle [\mathcal{H}_0\mathcal{H}_1], [\mathcal{H}_0\mathcal{H}_2] \rangle$ .  $\square$

**Theorem 12.** *Let  $\mathbb{K}_3 = \mathbb{k}(\sqrt{2}) = \mathbb{Q}(\sqrt{2}, \sqrt{p_1p_2}, i)$ , then*

- (1) *If  $q(\mathbb{K}_3^+/\mathbb{Q}) = 1$ , then four classes capitulate in  $\mathbb{K}_3$  and  $\ker J_{\mathbb{K}_3} = \langle [\mathcal{H}_0], [\mathcal{H}_1\mathcal{H}_2] \rangle$ .*
- (2) *If  $q(\mathbb{K}_3^+/\mathbb{Q}) = 2$ , then  $\ker J_{\mathbb{K}_3} = \langle [\mathcal{H}_0] \rangle$ .*

*Proof.* The results are deductions from Corollary 5.  $\square$

From Theorems 11, 12 we deduce the following result.

**Corollary 8.**  $\ker J_{\mathbb{k}^{(*)}} = \mathbf{C}_{\mathbb{k},2} = Am_s(\mathbb{k}/\mathbb{Q}(i))$ .

## REFERENCES

- [1] A. Azizi, *Unités de certains corps de nombres imaginaires et abéliens sur  $\mathbb{Q}$* , Ann. Sci. Math. Québec **23** (1999), no 1, 15-21.
- [2] A. Azizi, *Sur la capitulation des 2-classes d'idéaux de  $\mathbb{k} = \mathbb{Q}(\sqrt{2pq}, i)$ , où  $p_1 \equiv -q \equiv 1 \pmod{4}$* , Acta Arith. **94** (2000), 383-399.
- [3] A. Azizi, *Construction de la tour des 2-corps de classes de Hilbert de certains corps biquadratiques*, Pacific J. Math. **208** (2003), 1-10.
- [4] A. Azizi, *Sur les unités de certains corps de nombres de degré 8 sur  $\mathbb{Q}$* , Ann. Sci. Math. Québec **29** (2005), no 2, 111-129.
- [5] A. Azizi et M. Taous, *Détermination des corps  $\mathbf{k} = \mathbb{Q}(\sqrt{d}, \sqrt{-1})$  dont les 2-groupes de classes sont de type  $(2, 4)$  ou  $(2, 2, 2)$* , Rend. Istit. Mat. Univ. Trieste. **40** (2008), 93-116.
- [6] A. Azizi, A. Zekhnini et M. Taous, *On the generators of the 2-class group of the field  $\mathbf{k} = \mathbb{Q}(\sqrt{d}, i)$* , IJPAM, Volume **81**, No. 5 (2012), 773-784.
- [7] C. Chevalley, *Sur la théorie du corps de classes dans les corps finis et les corps locaux*, J. Fac. Sc. Tokyo, Sect. 1, t.**2**, (1933), 365-476.
- [8] F. Terada, *A principal ideal theorem in the genus fields*, Tohoku Math. J. **23**, No. 2 (1971), 697-718.
- [9] H. Furuya, *Principal Ideal Theorems in the Genus Field for Absolutely Abelian Extensions*, Journal of Number Theory **9**, (1977), 4-15.
- [10] H. Hasse, *Über die Klassenzahl abelscher Zahlkörper*, Berlin, Akademie-Verlag, (1952).
- [11] H. Wada, *On the class number and the unit group of certain algebraic number fields*, J. Fac. Univ. Tokyo Sect. I **13** (1966), 201-209.
- [12] M. Hirabayashi and K. Yoshino, *Unit indices of imaginary abelian number fields of type  $(2, 2, 2)$* , J. N. Theory, **34**, No. 3 (1990), 346-361.
- [13] S. Kuroda, *Über den Dirichletschen Körper*, J. Fac. Sci. Imp. Univ. Tokyo, sec I, vol IV, part **5**, (1943), 383-406.
- [14] T. M. McCall, C. J. Parry, R. R. Ranalli, *Imaginary bicyclic biquadratic fields with cyclic 2-class group*, J. Number Theory **53**, 88-99 (1995).
- [15] T. Kubota, *Über den bizyklischen biquadratischen Zahlkörper*, Nagoya Math. J. **10** (1956), 65-85.

ABDELMALEK AZIZI ET ABDELKADER ZEKHNINI: DÉPARTEMENT DE MATHÉMATIQUES, FACULTÉ DES SCIENCES, UNIVERSITÉ MOHAMMED 1, OUJDA, MOROCCO

*E-mail address:* `abdelmalekazizi@yahoo.fr`

*E-mail address:* `zekha1@yahoo.fr`

MOHAMMED TAOUS: DÉPARTEMENT DE MATHÉMATIQUES, FACULTÉ DES SCIENCES ET TECHNIQUES, UNIVERSITÉ MOULAY ISMAIL, ERRACHIDIA, MOROCCO

*E-mail address:* `taousm@hotmail.com`